Unlocking the Standard Model. IV. N=1 AND 2 generations of quarks: spectrum and mixing

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UNLOCKING THE STANDARD MODEL

IV . $N = 1$ AND 2 GENERATIONS OF QUARKS : SPECTRUM AND MIXING

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Abstract: Using a one-to-one correspondence between its complex Higgs doublet and very specific quadruplets of bilinear quark operators, the Glashow-Salam-Weinberg model for 2 generations is extended, without adding any extra fermion, to 8 composite Higgs multiplets. 8 is the minimal number required to suitably account, simultaneously, for the pseudoscalar mesons that can be built with 4 quarks and for the mass of the W gauge bosons. Their masses being used as input, together with elementary low energy considerations for the pions, we calculate all other parameters, masses and couplings. We focus in this work on the spectrum of the 8 Higgs bosons (which all potentially contribute to the W and quark masses), and on the mixing angles, leaving the study of couplings and their content of non-standard physics to a subsequent work.

We start with 1 generation, which already makes a 2-Higgs-doublet model. We show that the leptonic decays of charged pions are suitably described. One of the 2 Higgs bosons is extremely light, while the mass of the "quasi-standard" one is only $\sqrt{2}m_\pi$, showing the need for more generations. For 2 generations, we show that the $u-c$ and $d-s$ mixing angles, respectively θ_u and θ_d satisfy the relation $\tan(\theta_d+\theta_u)\tan(\theta_d-\theta_u) = \left(\frac{1}{m^2_{K^\pm}} - \frac{1}{m^2_{D^\pm}}\right) \Big/ \left(\frac{1}{m^2_{\pi^\pm}} - \frac{1}{m^2_{D_s^\pm}}\right)$. First, θ_u is set to 0, which allows the calculation ab initio of $\theta_d \approx \theta_c$ that we get 15% off its experimental value. Problems however remain, concerning in particular leptonic decays of π^+ and K^+. They are lifted in the second part of the study in which θ_u is determined to be close to $\sqrt{m_u/m_c}$.

The 8 Higgs bosons fall into one triplet, two doublets and one singlet. In the triplet stand three states with masses from $2.9\,GeV$ to $3.25\,GeV$. The singlet has a mass of $1.65\,GeV$. The masses of the last four neutral scalars should not exceed $90\,MeV$. We accordingly witness the emergence of light scalars. Hierarchies between vacuum expectation values, which are large for 1 generation, become much smaller for 2 generations. That the mass of (at least) one of the Higgs bosons grows like that of the heaviest $\bar q\gamma_5 q$ bound state is one of the hints that calls for a third generation of quarks.

Parameters are very fine-tuned. An example of this is the crucial role played by the small θ_u which cannot be safely switched to 0. In addition, several parameters turn out to have no trustable expansions in terms of small parameters like m_π, θ_u or θ_d. The sector of the light scalars is specially delicate to handle, such that definitive conclusions cannot truly be drawn with 2 generations only.

Inside the chiral $U(4)_L\times U(4)_R$ group, relevant symmetries and their breaking are investigated in detail. The generators of $U(1)_L$ or $U(1)_R$ swap the parity of fermion bilinears. Orthogonal to the chiral group $SU(2)_L\times SU(2)_R$, a second, similar group, the diagonal part of which flips generations, moves inside the 8-dimensional space of Higgs multiplets.

PACS: 02.20.Qs 11.15.Ex 11.30.Hv 11.30.Rd 11.40.Ha 12.15.Ff 12.60.Fr 12.60.Rc

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Chapter 1

Overview

1.1 Introduction

The Higgs boson of the Glashow-Salam-Weinberg (GSW) [1] model may be a fundamental scalar and the only one of this sort. There however exist in nature scalar mesons which are most probably quark-antiquark composites. It is thus natural to wonder whether the Higgs boson could be such a particle, that is, just one member of the family of scalar mesons.

Previous tentatives led to the introduction of super-heavy quarks (techniquarks) [2]. This was thought to be the only solution to the mismatch between the electroweak scale (the mass of the W or the vacuum expectation value of the Higgs boson) and the chiral scales m_π, f_π, that unavoidably led, otherwise, to $m_W \simeq m_\pi$.

We shall follow here an orthogonal way and only interpret this mismatch as the need for (at least) two different scales, and thus for (at least) 2 Higgs bosons with vacuum expectation values (VEV's) of the order of m_W and m_π. Instead of introducing extra fermions, we therefore prefer to introduce extra scalars, and we do it in a natural way.

Our definition of naturality is the simplest possible: all known particles, presently mesons and quarks [1] should be described in agreement with observations, including "the" Higgs boson (presumably the $125\,GeV$ state discovered at the LHC [3]). All parameters (VEV's, couplings) should be calculable in terms of physical quantities (masses of pseudoscalar mesons, W mass, quark masses). Their number we shall reduce from the start as much as possible by simple and sensible physical arguments, like the absence of coupling between scalar and pseudoscalar mesons, that of flavor changing neutral currents (FCNC), the need for 3 true Goldstone bosons to provide 3 longitudinal degrees of freedom for the W's, the role of Yukawa couplings to provide soft masses to the other "Goldstones" of the broken chiral symmetry *etc*. In this perspective, a model built to extend or complete the Standard Model should: first, not get in contradiction with present observations; secondly, be able to predict the properties and couplings of all states which have not been observed yet.

Along this path, one is unavoidably led to introduce several Higgs multiplets. The $2N$ quarks of N generations are the building blocks for $(2N)^2$ pseudoscalar mesons and $(2N)^2$ scalar mesons. The total of $8N^2$ such composite states should fit into $2N^2$ quadruplets (or complex doublets). In particular, for 1 generation, 2 Higgs multiplets are expected. They involve 4 pseudoscalar mesons (2 neutral and 2 charged), 2 Higgs bosons (neutral scalars) and 2 charged scalars.

This would only be phraseology without the one-to-one correspondence that we demonstrate, concerning the transformations by the weak group $SU(2)_L$, between the complex Higgs doublet of the GSW model and two sets of N^2 quadruplets of bilinear quark operators. The first set is made of quadruplets of the type $(\bar q_i q_j, \overrightarrow{\bar q_i \gamma_5 q_j})$ that

[1] though quarks are not particles.

is, one scalar and 3 pseudoscalars, and the second set of quadruplets of the type $(\bar q_i\gamma_5 q_j, \overrightarrow{\bar q_i q_j})$. All quadruplets transform alike by $SU(2)_L$ and each one includes both parities. The two sets are parity-transformed of each other.

Each quadruplet has to be normalized. The normalization factors must in particular make the transition between bilinear quark operators of dimension $[mass]^3$ and bosonic fields of dimension $[mass]$. Since each quadruplet includes one scalar $\bar q q$ operator with $<\bar q q>= \mu^3 \neq 0$, its natural normalization is realized through the factor

$$\frac{v}{\sqrt{2}\mu^3}. \tag{1.1}$$

In this way, the corresponding Higgs boson $\frac{v}{\sqrt{2}\mu^3}\bar q q$ gets a "bosonic" VEV $\frac{v}{\sqrt{2}}$ reminiscent of the GSW model. One normalizes all 4 elements of the same quadruplet by the same factor. Thus, to each quadruplet will be accordingly associated one "v" and one "μ^3", which makes, for 2 generations, a total of 8 "bosonic" VEV's and 8 "fermionic" VEV's. We shall suppose in the following that $<\bar q_i\gamma_5 q_j>=0$ and that $<\bar q_i q_j>=<\bar q_j q_i>$, though both statements can only be approximations in a theory that violates parity and also, eventually, C and CP.

The dual nature of the components of the Higgs quadruplets will be extensively used. The simple example below shows the principle of the method. Let a given quadruplet X (see (4.1)) include the charged pseudoscalar bilinear quark operator (after the normalization explained above has been implemented)

$$X^+ = \frac{v_X}{\sqrt{2}\mu^3}\frac{1}{\sqrt{2}}(2\bar u\gamma_5 d), \tag{1.2}$$

and, at the same time, a "Higgs boson" (scalar with non-vanishing VEV) $X^0 = \frac{v_X}{\sqrt{2}\mu^3}\frac{\bar u u+\bar d d}{\sqrt{2}}$ which has a VEV $\frac{<\bar u u+\bar d d>}{\sqrt{2}} = \mu^3$. After the mixing of d and s quarks has been accounted for, X^+ can be expressed in terms of quark mass states, which yields $X^+ = \frac{v_X}{\sqrt{2}\mu^3}\sqrt{2}(\cos\theta_c\bar u_m\gamma_5 d_m + \sin\theta_c \bar u_m\gamma_5 s_m)$. Now, PCAC [4] [5] for the $\pi^\pm$ mesons yields $i(m_u+m_d)\bar u_m\gamma_5 d_m = \sqrt{2}f_\pi m_{\pi^+}^2\pi^+$, in which π^+ is the charged pion (mesonic) interpolating field with dimension $[mass]$. This makes that, at least at low energy, one can also write $X^+ = \frac{v_X}{\sqrt{2}\mu^3}\sqrt{2}\left(\cos\theta_c\frac{-i\sqrt{2}f_\pi m_{\pi^+}^2}{m_u+m_d}\pi^+ + \ldots\right)$, which therefore now appears as a bosonic field like the components of the Higgs doublets of the GSW model. Eventually, one can also use the Gell-Mann-Oakes-Renner (GMOR) relation [6] [7] $(m_u+m_d)<\bar u_m u_m + \bar d_m d_m>= 2f_\pi^2 m_{\pi^+}^2$ to relate μ^3 to pionic parameters, which leads finally to

$$[X^+] = -i\frac{v_X}{f_\pi}(\cos\theta_c\pi^+ + \ldots). \tag{1.3}$$

In (1.3) we used the notation $[X^+]$ for X^+ when it is expressed in terms of a bosonic field. This notation we shall use throughout the paper: for any Higgs multiplet, Δ_i stands for its expression in terms of bilinear quark operators, and $[\Delta_i]$ stands for its "bosonic" form.

The calculations have been performed for 1 and 2 generations. It turns out that no simple argument or general principle could have anticipated the results, though they can be understood in simple terms a posteriori. It is also evident that a suitable solution cannot exist with a number of Higgs multiplets smaller that the one that we have introduced because, in particular, it could not fit observed pseudoscalar mesons. In this respect, the extension that we propose for the GSW model is minimal.

Two among its main features are the following:

* it is very fine-tuned;

* the $d-s$ and $u-c$ mixing angles θ_d and θ_u are independent parameters and, though $\theta_u \approx \sqrt{m_u/m_c} \ll 1$, it cannot be turned safely to 0.

1.2 Main results for 1 generation

Only 2 quarks (u,d) are present. They build up 4 pseudoscalar mesons (the 3 pions and the η flavor singlet) and 4 scalars. The latter include 2 neutral states which are 2 Higgs bosons, and 2 charged scalars. These 8 states fit into

2 quadruplets.

The W and π masses are inputs. The 3 longitudinal $W_\parallel$'s are built in this case from the 2 charged scalars and from the neutral pseudoscalar singlet; the three of them accordingly disappear from the physical spectrum.

The 2 Higgs bosons have respective masses $\sqrt{2}m_\pi \approx 197\,MeV$ and $m_\pi \frac{f_\pi}{\sqrt{2}m_W/g} \approx 69\,KeV$. They have the same ratio as the corresponding VEV's, respectively $\frac{f_\pi}{\sqrt{2}}$ and $\frac{\sqrt{2}m_W}{g}$, exhibiting a large hierarchy ≈ 2858.

PCAC provides the usual correspondence between pseudoscalar bilinear and pions and their leptonic decays through W's are suitably described.

Very light scalars start to spring out. For one generation there is only one such particle.

While $<\bar u u + \bar d d>$ is determined by the GMOR relation, one gets [2] $\hat r_X \equiv \frac{<\bar u u - \bar d d>}{<\bar u u + \bar d d>} = -\frac12 \frac{m_u+m_d}{m_u-m_d}$, which already points out at a negative d quark mass parameter $m_d = -|m_d|$ as will be confirmed for 2 generations. Then, $\hat r_X = \frac12 \frac{|m_d|-m_u}{|m_d|+m_u} \approx \frac16$.

The price to pay for this drastic truncation of the physical world is threefold:
* a very large hierarchy between the 2 bosonic VEV's;
* a very small mass $\sqrt{2}m_\pi$ for the heaviest Higgs boson which cannot be compared with the expected $125\,GeV$ [3] and could naively look like the revival of the mismatch between m_W and m_π that led to technicolor models;
* the disappearance of the singlet η pseudoscalar meson in favor of the neutral longitudinal $W^3_\parallel$.

These issues get on their way to a solution when one increases by 1 the number of generations. In particular, the mass of the "quasi-standard" Higgs boson becomes comparable to that of the heaviest pseudoscalar meson, D_s instead of π.

1.3 Main results for 2 generations

4 quarks (u, c, d, s) are now involved, which build up 32 $\bar q_i q_j$ and $\bar q_i \gamma_5 q_j$ composite states. These fit into 8 quadruplets. There are therefore in particular 8 Higgs bosons.

There are 2 mixing angles: θ_u describes the mixing between u and c flavor eigenstates while θ_d concerns d and s. Flavor and gauge symmetries are tightly entangled in this extension and the freedom to tune θ_u to 0 by a flavor rotation no longer exists. This makes that that the Cabibbo angle $\theta_c \equiv \theta_d - \theta_u$ cannot describe alone correctly the physics under concern. These features are exhibited by studying successively the case when one approximates θ_u to 0 and the one when both $\theta_d \neq 0$ and $\theta_u \neq 0$.

1.3.1 The case $\theta_d \neq 0, \theta_u = 0$

The 8 Higgs bosons split into 1 triplet, 2 doublets and 1 singlet. Inside each of these, they are close to degeneracy. 3 have masses $\approx \sqrt{2}m_{D_s}$, more precisely $2.79\,GeV, 2.796\,GeV, 2.80\,GeV$, 2 have intermediate masses $1.23\,GeV, 1.26\,GeV$, 1 has a very small mass $19\,MeV$ and the last two only get massive by quantum corrections.

The hierarchies between VEV's stay below 151 (instead of 2858 for 1 generation).

The situation has improved a lot with respect to 1 generation; indeed, the masses of the quasi-standard Higgs boson(s) suitably increase and depart from m_π and the hierarchies between VEV's go down to more reasonable values.

[2] We use the same notations as in the bulk of the paper.

θ_d, that one identifies with the Cabibbo angle is expressed by the 2 formulæ

$$\tan^2\theta_d = \frac{1/m^2_{K^+} - 1/m^2_{D^+}}{1/m^2_{\pi^+} - 1/m^2_{D_s}} \approx \frac{m^2_{\pi^+}}{m^2_{K^+}}\Big(1 - \frac{m^2_{K^+}}{m^2_{D^+}} + \frac{m^2_{\pi^+}}{m^2_{D_s}}\Big) + \mathcal{O}\left(\Big(\frac{m^2_\pi}{m^2_{K,D,D_s}}\Big)^2\right),$$

$$\tan^2\theta_d = \frac{|m_d| + m_u}{m_s - m_u} \approx \frac{|m_d|}{m_s}, \quad m_d = -|m_d| < 0. \tag{1.4}$$

The first equation in (1.4), yields

$$\theta_d \approx .26685, \tag{1.5}$$

15 % off the experimental value of the Cabibbo angle

$$\theta_c^{exp} \approx .22759 \simeq \sqrt{\frac{|m_d|}{m_s}}. \tag{1.6}$$

A negative sign for m_d is needed, like for 1 generation. Since $|m_d| > m_u$, it yields, by the GMOR relation, a negative sign for $< \bar u u + \bar d d >= 2f_\pi^2 m_\pi^2/(m_u + m_d)$.

One however still faces problematic issues :

• the nice description of π^+ leptonic decays that we had found for 1 generation gets totally spoiled; the situation could only be improved if b_Ω was very small;

• taking the masses of the charged pseudoscalar mesons as inputs, the mass of the neutral K mesons is off by $140\,MeV$, unless one goes to a value larger than 1 for $\hat b_X \equiv (\hat v_X/\hat v_H)^2$, in conflict with most needed orthogonality relations;

• defining its interpolating field as proportional to $\bar u\gamma_5 u + \bar d\gamma_5 d$, the η meson cannot be set orthogonal to $K^0 + \bar K^0$ (actually, we do not get too worried by this problem because of the mixing between neutral pseudoscalars);

• 2 ratios of bosonic VEV's, $b_\Omega \equiv (v_\Omega/\hat v_H)^2$ and $\hat b_\Omega = (\hat v_\Omega/\hat v_H)^2$ come out too large to match intuitive arguments concerning (non-diagonal) quark condensates;

• the last problem concerns mixing, and proves later to be correlated with the previous one. On one side eqs. (1.4) give fairly good estimates of the mixing angle; the result is independent of the so-called b parameters (ratios of bosonic VEV's) and looks robust. On another side, Yukawa couplings provide diagonal and non-diagonal mass terms for the d and s quarks: with intuitive notations

$$\tan 2\theta_c = -\frac{2\mu_{ds}}{\mu_d - \mu_s}, \tag{1.7}$$

in which μ_{ds}, μ_d, μ_s depend on the b parameters through the normalizing coefficients (1.1) of the 8 Higgs quadruplets. The paradox is that, at the values of the b parameters which fit all other data, in particular pseudoscalar meson masses, μ_{ds} comes very close to a pole, like if the "fermionic mixing angle" was close to maximal ($\pi/4$). So, either quark mixing exhibits a dual nature (maximal mixing or close to being at present only known for leptons), or one must find a way out of this paradox. It would be feasible at very small values of the parameters b_Ω and $\hat b_\Omega$, which seems excluded at $\theta_u = 0$.

1.3.2 The case $\theta_d \neq 0, \theta_u \neq 0$

The first equation in (1.4) is only the approximation at $\theta_u = 0$ of the exact formula

$$\boxed{\tan(\theta_d + \theta_u)\tan(\theta_d - \theta_u) = \frac{\dfrac{1}{m^2_{K^\pm}} - \dfrac{1}{m^2_{D^\pm}}}{\dfrac{1}{m^2_{\pi^\pm}} - \dfrac{1}{m^2_{D_s^\pm}}}} \tag{1.8}$$

which shows that θ_d and θ_u cannot be dealt with independently. Using the experimental value of Cabibbo angle (1.6), (1.8) yields

$$\boxed{\theta_u \approx .04225, \quad \theta_d \approx .2698} \tag{1.9}$$

The values that we find for the mixing angles correspond to a good approximation to [3] θ_d $(\approx \theta_d - \theta_u) \sim \sqrt{|m_d|/m_s} \approx .2236$ and $\theta_u \sim \sqrt{m_u/m_c} \approx .044$ [4].

Despite its very small value, switching on θ_u has very important consequences, for example on the values of the b parameters, and brings a very good agreement between the model and the basis of meson physics :

• leptonic decays of π^+ and K^+ are well described;

• the parameters b_Ω and $\hat{b}_\Omega$ become very small which opens the way to a matching between bosonic and fermionic mixing;

• the masses of neutral pion and kaon are now well accounted for, and the D^0 is only off by $20\,MeV$.

The spectrum of Higgs bosons is changed to $m_{\hat{H}^3} \approx 3.24\,GeV, m_{H^0} \approx 1.65\,GeV, m_{X^0} \approx 3.24\,GeV, m_{\Omega^0} \approx 86\,MeV$.

Ξ^0 and $\hat{\Xi}^3$ are still expected to be very light, and so does $\hat{\Omega}^3$ because $\hat{b}_\Omega$ is expected to be of the same order of magnitude as b_Ω. $\hat{b}_X = \mathcal{O}(1)$ is preferred, though it is difficult to give yet a precise value. The positive improvement is that it does not need any longer to be larger than 1.

All parameters are very fine tuned. The importance of the small θ_u is just one among the symptoms of this; one often deals with rapidly varying functions which furthermore have poles, parameters that have no trustable expansions at the chiral limit, "unlucky" coincidences *etc*. It is probably the price to pay for naturalness: it is indeed very unlikely that some general principle or god-given symmetry miraculously tunes the values of physical observables up to many digits after the decimal dot. Nature is obviously fine tuned and a model that pretends to describe it accurately has many chances to be fine-tuned, too.

1.4 Principle of the method

One works at two levels, bosonic and fermionic.

• Bosonic considerations rely on few statements.

⋆ The mass of the W gauge bosons, which, in this framework, comes from the VEV's of several Higgs bosons, is known.

⋆ The masses of all charged pseudoscalar mesons is also known with high precision. One should be more careful about some neutral pseudoscalars that can mix and the definition of which in terms of quark bilinears can be unclear.

⋆ The effective Higgs potential to be minimized is built from the genuine scalar potential, suitably chosen, to which is added the bosonised form of the Yukawa couplings. Its minima are constrained to occur at the set of bosonic VEV's $v/\sqrt{2}$'s.

⋆ The VEV's are supposed to be real and, therefore, there squares to be positive.

⋆ Among the components of the Higgs 8 quadruplets:

* there must exist 3 true Goldstones related to the breaking of the local $SU(2)_L$;

* all other scalar and pseudoscalar fields that do not have non-vanishing VEV's are pseudo-Goldstone bosons that

[3] We take $m_u = 2.5\,MeV, |m_d| = 5\,MeV, m_s = 100\,MeV, m_c = 1.2755\,GeV$.

[4] Among first attempts to calculate the Cabibbo angle are the ones by Oakes [8] and by Weinberg [9]. Since then, it has been a most sought for goal of calculating the mixing angles from basic principle (see for example [10] and [11] in which specific hypotheses are made concerning the symmetries involved and/or the mass matrices, and the estimate for $\theta_d - \theta_u$ in [12] based on the sole existence of mass hierarchies among quarks).

get "soft" masses via the Yukawa couplings at the same time as quarks get massive. This restricts and simplifies the scalar potential.

$\star$ The $mass^2$ of the known pseudoscalar mesons will be calculated as the ratios of the corresponding quadratic terms in the bosonised Yukawa Lagrangian and in the kinetic terms. They depend one the VEV's, on the mixing angle(s), and of course on the set of Yukawa couplings. Their number is reduced by a suitable and motivated choice for the Yukawa potential.

$\star$ Additional relations among Yukawa couplings arise from various sets of constraints:
* no transition should occur between scalar and pseudoscalar states;
* likewise, no transition should occur between charged pseudoscalar mesons;
* similar orthogonality relations are explored among neutral pseudoscalars and, for 2 generations, most of them (but not all of them) can be satisfied.

• Fermionic considerations use the genuine (not bosonised) form of the Yukawa Lagrangian, which provides mass terms for the 4 quarks, both diagonal and non-diagonal. We mainly use them at $\theta_u = 0$.

$\star$ A first set of constraints comes when turning to 0 the mixing between the u and c quarks; then the mixing angle θ_d becomes the Cabibbo angle θ_c;
$\star$ A second set of constraints comes from requirements of reality for the quarks masses;
$\star$ A third set of constraints comes when studying the s quark at the chiral limit $m_u, m_d \to 0$.

All these constraints are checked by evaluating the masses of pseudoscalar mesons, the leptonic decays of charged pseudoscalars ... and used to predict unknown quantities in particular the masses of the scalar mesons (Higgs bosons).

1.5 Changes between version 2 (this version) and version 1 of this work

Since the versions 1 and 2 have the same arXiv number, we list here the differences between the two.

The main difference concerns the extension to $\theta_u \neq 0$. It was indeed realized that, for 2 generations, assuming $\theta_d \neq 0$ and $\theta_u = 0$ led to grossly incorrect leptonic decays of π^+ and K^+. The situation was paradoxical since no problem arose for 1 generation.

Version 2 includes accordingly the study of the 3 cases: 1 generation (it was not present in version 1 but had to be included here to show that leptonic decays were correctly described, and also to unify the notations between [13][14] and the case of 2 generations), 2 generations with $\theta_d \neq 0, \theta_u = 0$ and 2 generations with $\theta_d \neq 0, \theta_u \neq 0$. It is shown how both "chiral scales" and weak scales can be accounted for without having to introduce extra super-heavy fermions. In relation with this, leptonic decays are investigated with special care.

θ_u is determined to be very close to $\sqrt{m_u/m_c}$, which is a very small number. Nevertheless, it has a crucial importance and cannot be safely tuned to 0, unlike in the Glashow-Salam-Weinberg model. The values of several parameters and the spectrum of Higgs bosons get modified, which shows that the underlying physics is very fine tuned. We give examples of this and insist on the fact that several parameters of the model lie in regions of rapid variation, eventually close to poles, that others don't have reliable expansions in terms of small parameters like the pion mass, θ_d or θ_u ...

The new values of some parameters obtained at $\theta_u \neq 0$ relieve tensions that arose when $\theta_u = 0$, in particular concerning the parameter $\hat{b}_X$ and the masses of π^0, K^0 and D^0.

There are now 12 figures instead of 3.

Many remarks, footnotes and 1 appendix have been added to guide the reader, such that all equations in the core of the paper can be easily reproduced.

Several references have been added.

Misprints have been corrected. Misplaced parentheses in eq.(124) of version 1 were the most important. Fortunately, this amounted to replace $cos^2(2\theta_d)$ with 1 for some contributions to the masses of neutral pseudoscalar mesons, which is numerically small (numbers have been corrected). Modifications to analytical expressions, for example eq. (127) of version 1 have of course been done.

1.6 Contents

• **Chapter 2** is dedicated to general considerations.

* Section 2.1 establishes, in the general case of N generations, a one-to-one relation between the complex Higgs doublet of the Glashow-Salam-Weinberg model and $2N^2$ very specific quadruplets of bilinear $\bar q_i q_j$ and $\bar q_i \gamma_5 q_j$ quark operators including either 1 scalar and 3 pseudoscalars, or 1 pseudoscalar and 3 scalars. To this purpose, the group $SU(2)_L$ of weak interactions is trivially embedded into the chiral group $U(2N)_L \times U(2N)_R$.

The normalization of the quadruplets is then explained, which introduces $2N^2$ "bosonic" VEV's of the form $v/\sqrt{2}$ and $2N^2$ "fermionic" VEV's which are $< \bar q q >$ condensates.

The connection is made between parity and the 2 generators $\mathbb{I}_L$ or $\mathbb{I}_R$.

* Section 2.2 presents general considerations concerning the Yukawa couplings $\mathcal{L}_{Yuk}$. Arguments will be given concerning how and why they can be simplified. They are chosen as the most straightforward generalization to N generations of the most general Yukawa couplings for 1 generation, in which 2 quarks are coupled to 2 Higgs doublets. Yukawa couplings are no longer passive in determining the VEV's of the Higgs bosons. This leads to introduce their bosonised form. Subtracting it from the scalar potential yields an effective potential that can be used to find the (bosonic) VEV's of the Higgs bosons. A first set of constraints is established by the condition that no transitions should occur between scalars and pseudoscalar mesons.

* Section 2.3 presents and motivates our simple choice for the Higgs potential. The minimization of the corresponding effective potential (see above) leads to another set of relations between its parameters, Yukawa couplings and bosonic VEV's. Goldstones and pseudo-Goldstones are investigated, in relation with the concerned broken symmetries. The (soft) masses of the pseudo-Goldstones can be calculated from the bosonised form of the Yukawa Lagrangian.

* Section 2.4 gives general formulæ for the masses of the Higgs bosons.

• **Chapter 3** deals with the simplest case of 1 generation. Using as input the masses of the W's, pions, u and d quarks, bosonic and fermionic equations are solved which yield the spectrum of the 2 Higgs bosons, the values of the 4 VEV's and all couplings. The leptonic decays of π^+ are shown to be in agreement with the usual PCAC estimate. The content of this section overlaps with [13] and [14]. However, the notations are unified with the ones of the 2-generation case for easier comparison.

• **Chapter 4** gives general results in the case of 2 generations.

* Section 4.1 displays the 8 Higgs quadruplets. The choice of the quadruplet that contains the 3 Goldstones of the spontaneously broken $SU(2)_L$ is motivated. Notations that will be used throughout the paper are given.

* Sections 4.2 and 4.3 give generalities concerning the kinetic terms, Yukawa couplings and the Higgs potential. The mass of the W's is expressed in terms of the bosonic VEV's.

* Section 4.7 identifies the group of transformations that moves inside the space of quadruplets. Its generators commute with the ones of the gauge group.

* Section 4.4 is devoted to charged pseudoscalar mesons. Their masses and orthogonality relations are explicitly written.

* Section 4.5 shows how the formula (1.8) relating θ_d, θ_u and the masses of charged pseudoscalar mesons is obtained very simply. It does not depend on low energy theorems like GMOR (which are badly verified for heavy mesons) and only relies on the statement that $\bar u_m\gamma_5 d_m, \bar u_m\gamma_5 s_m, \bar c_m\gamma_5 d_m, \bar c_m\gamma_5 s_m$ are proportional to the interpolating fields of, respectively π^+, K^+, D^+, D_s^+. There is no need to know the proportionality constants, which makes this result specially robust. This section overlaps with [15], but adds some new features and strengthens it.

* Section 4.6 gives the general formulæ for the masses and orthogonality conditions for π^0, η, K^0 and D^0.

• **Chapter 5** deals with the approximation $\theta_u = 0$, keeping of course $\theta_d = \theta_c \neq 0$.

* In section 5.1, the value of the Cabibbo angle θ_c is extracted from the general formula (1.8). Its value falls within 15% of the experimental θ_c^{exp}.

* In section 5.2, we write a basic set of equations that will determine the b ratios of bosonic VEV's, and we show how, from the sole spectrum of charged pseudoscalar mesons, one already gets a lower bound on the mass of the "quasi-standard" Higgs boson $m_{\hat H^3} \geq \sqrt{2} m_{D_s}$.

* Section 5.3 is dedicated to neutral pseudoscalar mesons and to the constraints given by their masses and orthogonality. We find a tension concerning $\hat b_X = (<\hat X^3> / <X^0>)^2$, because its value obtained from the mesonic mass spectrum is slightly larger than 1 is in contradiction with orthogonality relations of K^0 to $\bar K^0$, and of π^0 to $K^0 + \bar K^0$.

* Section 5.4 studies charged scalar mesons. Their orthogonality relations cannot be satisfied unless they align with flavor eigenstates. This is not surprising since the two of them which coincide with the charged Goldstone bosons of the broken $SU(2)_L$ gauge symmetry are by construction flavor eigenstates.

* Section 5.5 summarizes all the bosonic constraints.

* In section 5.11, we study the masses of π^0, K^0 and D^0. In particular, $\hat b_X > 1$ is needed to correctly account for the mass of K^0; the latter is otherwise off by $140\, MeV$.

The next 3 sections deal with fermionic constraints.

* Section 5.6 lists the equations coming from the definition of quark masses in terms of Yukawa couplings and VEV's. Additional constraints are given by using the freedom (as we already did for bosons) to turn θ_u to 0.

* Section 5.7 displays the constraints coming from the reality of the quark masses. Among the outcomes are: - the knowledge of $|\hat r_H| = \left|\frac{<\bar c c - \bar s s>}{<\bar u u + \bar d d>}\right|$; - the expression of $\tan^2\theta_c$ in terms of quark masses as given by the second line of (1.4), which requires in particular $m_d < 0$.

* In section 5.8 we calculate the "fermionic mixing angle" from Yukawa couplings and show that it tends to be maximal, in contrast with the small value of the Cabibbo angle.

* Section 5.9 studies m_s at the chiral limit $m_u, m_d \to 0$. This determines in particular the sign of $\hat r_H$ and the value of the mass of the "quasi-standard" Higgs bosons $\hat H^3$.

* Section 5.10 summarizes the solution of all equations. It is shown how Ξ^0 and $\hat\Xi^3$, which are classically massless, are expected to get soft masses from quantum corrections. Hierarchies between VEV's are shown to be much smaller than for 1 generation.

* Section 5.12 is dedicated to leptonic decays of π^+ and K^+. We show that they cannot be suitably described for $\theta_d \neq 0$ and $\theta_u = 0$. The situation is therefore, at the moment, worse than for 1 generation.

* Section 5.13 is a brief summary of the case $\theta_u = 0$, mainly pointing at the problems that arise.

• **Chapter 6** concerns the general case $\theta_u \neq 0$ and $\theta_d \neq 0$.

* In section 6.1, we use the experimental value of the Cabibbo angle to calculate θ_u, which is very close to $\sqrt{m_u/m_c}$.

* In section 6.2 we re-analyze the leptonic decays of the charged pions and kaons. Unlike at $\theta_u = 0$, a nice agreement can be obtained. Then, the parameters of the model are updated, a large set of them being very sensitive to θ_u. In particular, b_Ω, and presumably $\hat{b}_\Omega$ too, are now very small.

* In section 6.3 we re-investigate the masses of π^0, K^0 and D^0. We find that, even for $\hat{b}_X \leq 1$, they can now be quite well accounted for. The D^0 is the worst, but its mass is only off by $20\,MeV$. One however needs a fairly large value of the $< \bar{c}c >$ condensate, which coincides with what we already suspected at $\theta_u = 0$ namely that, in one way or another, "some" heavy quark should have a large condensate.

* In section 6.4, we update the Higgs spectrum. The masses of the 3 heaviest Higgs bosons $\hat{H}^3, X^0, \hat{X}^3$ has increased to $2.9 - 3.2\,GeV$, H^0 has risen to an intermediate mass of $1.65\,GeV$ and the 4 others are light (they should not exceed $90\,MeV$).

* Section 6.6 concludes the case $\theta_d \neq 0, \theta_u \neq 0$. The tensions that occurred at $\theta_u = 0$ have been mostly removed or on their way to be (like the paradox of the fermionic "maximal mixing"). Including the 3rd generation of quarks is of course highly wished for, but goes technically largely beyond the limits of this work. We emphasize the impressive ability of this multi-Higgs model to account for the physics of both the broken weak symmetry and that of mesons. We also largely comment on the very fined-tuned character of all the physical outputs. Their sensitivity to the small θ_u is just one example among a list of parameters which have no trustable expansions at the chiral limit, or at the limit of small mixing angles.

• **Chapter 7** is a general conclusion, mostly focused on symmetries

* In section 7.1 we study how the chiral/gauge group $SU(2)_L \times SU(2)_R$ acts inside each quadruplet, which shows that the third generator T^3 of the custodial $SU(2)$ is identical to the electric charge [5].

We then study which generators of the diagonal $U(4)$ annihilate the Higgs states, which provides the properties of invariance of the vacuum.

Next, we show that this extension of the GSW model can be a right-handed gauge $SU(2)_R$ theory as well as a left-handed $SU(2)_L$ one, and that it is in principle ready to be a left-right gauge symmetry. This requires 6 true Goldstone bosons, which cannot be achieved with only 2 generations because some states should be absent from the physical spectrum which have in reality been observed.

Then, we make some more remarks concerning parity and its breaking.

We study the "generation" chiral group of transformations and show how the 8 Higgs quadruplets fall into 4 doublets of $SU(2)^g_R$ or $SU(2)^g_L$, and 1 triplet + 1 singlet of its diagonal $SU(2)^g$ subgroup.

* In section 7.2, we explain why the spectrum of the 8 Higgs bosons, 1 triplet, 2 doublets and 1 singlet can be considered to fall into representations of this generation $SU(2)^g$ subgroup orthogonal to the custodial $SU(2)$.

* In section 7.3 we make miscellaneous remarks and give prospects for forthcoming works. We emphasize the very important role of the normalization of bosonic asymptotic states to determine their couplings to quarks; we outline in particular why present bounds on the masses of light scalars have to be revised. We give a list of topics to be investigated and their spreading to other domains of physics.

• in **appendix** .1, we collate the expressions of bilinear flavor quark operators in terms of their mass counterparts and of the mixing angles θ_u and θ_d. These formulæ are used throughout the paper.

Note : to help the reader, formulæ which are valid for the whole paper, including definitions, and final results have been boxed.

[5] This had already been noticed in [16].

Chapter 2

General results

Embedding the gauge group $SU(2)_L$ into the chiral group $U(2N)_L \times U(2N)_R$, we start by establishing a one-to-one correspondence between the Higgs doublet of the GSW model and quadruplets of bilinear quark operators. We then proceed to constructing our multi-Higgs extension of the standard model. We introduce Yukawa couplings, then the genuine and effective scalar potentials. The latter plays an important role because the Yukawa couplings are no longer passive in defining the vacuum of the theory. Last we give general formulæ for the masses of the Higgs bosons.

2.1 A one-to-one correspondence

2.1.1 The Higgs doublet of the GSW model

We give below the laws of transformations of the components of the Higgs doublet of the GSW model, and, by a very simple change of variables, put them in a form that matches the ones of specific bilinear fermion operators that we shall introduce later.

The generators of the group $SU(2)_L$ are the three hermitian 2×2 matrices

$$\vec{T}_L = \frac{\vec{\tau}}{2}, \tag{2.1}$$

where the τ's are the Pauli matrices

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.2}$$

The Higgs doublet H is generally written

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi^1 + i\chi^2 \\ \chi^0 - \chi^3 \end{pmatrix}, \quad \chi^3 = ik^3, \tag{2.3}$$

in which $\chi^{0,1,2}$ and k^3 are considered to be real. The vacuum expectation value of H arises from $< \chi^0 >= v$ such that $< H >= \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$. H is in the fundamental representation of $SU(2)_L$ such that generators $\vec{T}_L$ act by

$$T^i_L.H = T^i_L H. \tag{2.4}$$

The transformed $T^i_L.\chi^\alpha, i = 1, 2, 3, \alpha = 0, 1, 2, 3$ of the components χ^α are naturally defined by

$$T^i_L.H = \frac{1}{\sqrt{2}} \begin{pmatrix} T^i_L.\chi^1 + iT^i_L.\chi^2 \\ T^i_L.\chi^0 - T^i_L.\chi^3 \end{pmatrix}, \tag{2.5}$$

such that the law of transformation (2.4) is equivalent to

$$\begin{aligned}
&T_L^1\cdot\chi^0=+\tfrac{i}{2}\,\chi^2,\ \ T_L^2\cdot\chi^0=+\tfrac{i}{2}\,\chi^1,\ \ T_L^3\cdot\chi^0=+\tfrac{1}{2}\,\chi^3,\\
&T_L^1\cdot\chi^1=-\tfrac{1}{2}\,\chi^3,\ \ T_L^2\cdot\chi^1=-\tfrac{i}{2}\,\chi^0,\ \ T_L^3\cdot\chi^1=+\tfrac{i}{2}\,h^2,\\
&T_L^1\cdot\chi^2=-\tfrac{i}{2}\,\chi^0,\ \ T_L^2\cdot\chi^2=+\tfrac{1}{2}\,\chi^3,\ \ T_L^3\cdot\chi^2=-\tfrac{i}{2}\,\chi^1,\\
&T_L^1\cdot\chi^3=-\tfrac{1}{2}\,\chi^1,\ \ T_L^2\cdot\chi^3=+\tfrac{1}{2}\,\chi^2,\ \ T_L^3\cdot\chi^3=+\tfrac{1}{2}\,\chi^0.
\end{aligned}\tag{2.6}$$

It takes the form desired for later considerations

$$\begin{aligned}
T_L^i\,.\,h^j &= -\tfrac{1}{2}\left(i\,\epsilon_{ijk}h^k+\delta_{ij}\,h^0\right),\\
T_L^i\,.\,h^0 &= -\tfrac{1}{2}\,h^i,
\end{aligned}\tag{2.7}$$

when one makes the substitutions

$$\chi^0=-h^3,\quad \chi^1=h^1,\quad \chi^2=-h^2,\quad \chi^3=h^0\quad (\Leftrightarrow k^3=-ih^0).\tag{2.8}$$

H then rewrites

$$H=\frac{1}{\sqrt{2}}\begin{pmatrix} h^1-ih^2\\ -(h^0+h^3)\end{pmatrix},\tag{2.9}$$

and $<\chi^0>=v$ is thus tantamount to $<h^3>=-v$.

Later, we shall often, instead of complex Higgs doublets, consider indifferently quadruplets, for example, in this case

$$H=\frac{1}{\sqrt{2}}(h^0,h^3,h^+,h^-),\quad h^\pm=h^1\pm i\,h^2,\tag{2.10}$$

keeping in mind that, to any such quadruplet is associated a complex doublet in the fundamental representation of $SU(2)_L$ given by (2.9).

2.1.2 Embedding the gauge group into the chiral group

For N generations of quarks, that is, $2N$ quarks, let us embed $SU(2)_L$ into the chiral group $U(2N)_L\times U(2N)_R$ by representing its three generators as the following $2N\times 2N$ matrices

$$T^3=\frac{1}{2}\left(\begin{array}{c|c}\mathbb{I} & \\ \hline & -\mathbb{I}\end{array}\right),\quad T^+=T^1+iT^2=\left(\begin{array}{c|c} & \mathbb{I}\\ \hline & \end{array}\right),\quad T^-=T^1-iT^2=\left(\begin{array}{c|c} & \\ \hline \mathbb{I} & \end{array}\right),\tag{2.11}$$

in which $\mathbb{I}$ is the $N\times N$ identity matrix. They act trivially on $2N$ vectors of flavor quark eigenstates $\psi=(u,c,t,\ldots,d,s,b,\ldots)^t$ [1].

2.1.3 Quadruplets of bilinear quark operators

$\mathbb{M}$ being any $2N\times 2N$ matrix, we now consider bilinear quark operators of the form $\bar\psi\mathbb{M}\psi$ and $\bar\psi\gamma_5\mathbb{M}\psi$. $\mathcal{U}_R$ and $\mathcal{U}_L$ being transformations of $SU(2)_R$ and $SU(2)_L$ respectively, these bilinears transform by the chiral group according to

$$\begin{aligned}
(\mathcal{U}_L\times\mathcal{U}_R)\,.\,\bar\psi\frac{1+\gamma_5}{2}\mathbb{M}\psi &= \bar\psi\,\mathcal{U}_L^{-1}\,\mathbb{M}\,\mathcal{U}_R\,\frac{1+\gamma_5}{2}\psi,\\
(\mathcal{U}_L\times\mathcal{U}_R)\,.\,\bar\psi\frac{1-\gamma_5}{2}\mathbb{M}\psi &= \bar\psi\,\mathcal{U}_R^{-1}\,\mathbb{M}\,\mathcal{U}_L\,\frac{1-\gamma_5}{2}\psi.
\end{aligned}\tag{2.12}$$

[1]The superscript $(\ )^t$ means "transpose".

Writing $\mathcal{U}_R$ and $\mathcal{U}_L$ as

$$\mathcal{U}_{L,R} = e^{-i\alpha_i T^i_{L,R}}, \quad i = 1,2,3, \tag{2.13}$$

eq. (2.12) entails

$$\begin{aligned}
T^j_L \,.\, \bar\psi \mathbb{M}\psi &= -\frac{1}{2}\,\left(\bar\psi\,[T^j,\mathbb{M}]\,\psi + \bar\psi\,\{T^j,\mathbb{M}\}\,\gamma_5\psi\right),\\
T^j_L \,.\, \bar\psi \mathbb{M}\gamma_5\psi &= -\frac{1}{2}\,\left(\bar\psi\,[T^j,\mathbb{M}]\,\gamma_5\psi + \bar\psi\,\{T^j,\mathbb{M}\}\,\psi\right),\\
T^j_R \,.\, \bar\psi \mathbb{M}\psi &= -\frac{1}{2}\,\left(\bar\psi\,[T^j,\mathbb{M}]\,\psi - \bar\psi\,\{T^j,\mathbb{M}\}\,\gamma_5\psi\right),\\
T^j_R \,.\, \bar\psi \mathbb{M}\gamma_5\psi &= -\frac{1}{2}\,\left(\bar\psi\,[T^j,\mathbb{M}]\,\gamma_5\psi - \bar\psi\,\{T^j,\mathbb{M}\}\,\psi\right),
\end{aligned} \tag{2.14}$$

in which [,] and { , } stand respectively for the commutator and anticommutator.

Let us now define the specific $2N \times 2N$ matrices

$$\mathbb{M}^0 = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & M \end{array}\right), \mathbb{M}^3 = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & -M \end{array}\right), \mathbb{M}^+ = 2\left(\begin{array}{c|c} 0 & M \\ \hline 0 & 0 \end{array}\right), \mathbb{M}^- = 2\left(\begin{array}{c|c} 0 & 0 \\ \hline M & 0 \end{array}\right), \tag{2.15}$$

in which $\mathbb{M}^\pm = \mathbb{M}^1 \pm i\mathbb{M}^2$, and M is a real $N \times N$ matrix. Let us call (a^0, a^3, a^+, a^-) the generic components of the two sets of N^2 quadruplets

$$\bar\psi\Big(\mathbb{M}^0, \gamma^5\mathbb{M}^3, \gamma^5\mathbb{M}^+, \gamma^5\mathbb{M}^-\Big)\psi, \tag{2.16}$$

of the type $(\mathfrak{s}^0, \vec{\mathfrak{p}})$, made with one scalar and three pseudoscalars, and

$$\bar\psi\Big(\gamma^5\mathbb{M}^0, \mathbb{M}^3, \mathbb{M}^+, \mathbb{M}^-\Big)\psi, \tag{2.17}$$

of the type $(\mathfrak{p}^0, \vec{\mathfrak{s}})$, made with one pseudoscalar and three scalars. By (2.14), the a^i's transform by $SU(2)_L$ and $SU(2)_R$ according to

$$\boxed{\begin{array}{rcl} T^i_L \,.\, a^j &=& -\frac{1}{2}\left(i\,\epsilon_{ijk}a^k + \delta_{ij}\,a^0\right)\\ T^i_L \,.\, a^0 &=& -\frac{1}{2}\,a^i \end{array}} \tag{2.18}$$

or, equivalently, since it is often convenient to manipulate states with given electric charge,

$$\begin{array}{llll}
T^3_L.a^0 = -\frac{1}{2}a^3, & T^3_L.a^3 = -\frac{1}{2}a^0, & T^3_L.a^+ = -\frac{1}{2}a^+, & T^3_L.a^- = +\frac{1}{2}a^-,\\
T^+_L.a^0 = -\frac{1}{2}a^+, & T^+_L.a^3 = +\frac{1}{2}a^+, & T^+_L.a^+ = 0, & T^+_L.a^- = -a^0 - a^3,\\
T^-_L.a^0 = -\frac{1}{2}a^-, & T^-_L.a^3 = -\frac{1}{2}a^-, & T^-_L.a^+ = -a^0 + a^3, & T^-_L.a^- = 0,
\end{array} \tag{2.19}$$

and

$$\boxed{\begin{array}{rcl} T^i_R \,.\, a^j &=& -\frac{1}{2}\left(i\,\epsilon_{ijk}a^k - \delta_{ij}\,a^0\right)\\ T^i_R \,.\, a^0 &=& +\frac{1}{2}\,a^i \end{array}} \tag{2.20}$$

or, equivalently

$$\begin{array}{llll}
T^3_R.a^0 = +\frac{1}{2}a^3, & T^3_R.a^3 = +\frac{1}{2}a^0, & T^3_R.a^+ = -\frac{1}{2}a^+, & T^3_R.a^- = +\frac{1}{2}a^-,\\
T^+_R.a^0 = +\frac{1}{2}a^+, & T^+_R.a^3 = +\frac{1}{2}a^+, & T^+_R.a^+ = 0, & T^+_R.a^- = +a^0 - a^3,\\
T^-_R.a^0 = +\frac{1}{2}a^-, & T^-_R.a^3 = -\frac{1}{2}a^-, & T^-_R.a^+ = +a^0 + a^3, & T^-_R.a^- = 0.
\end{array} \tag{2.21}$$

The laws of transformations (2.7) and (2.18) being identical, we have therefore found $2N^2$ quadruplets isomorphic, for their law of transformation by $SU(2)_L$, to the complex Higgs doublet of the GSW model. We shall deal later with their normalizations.

The two sets respectively of the $(\mathfrak{s}^0, \vec{\mathfrak{p}})$ type and of the $(\mathfrak{p}^0, \vec{\mathfrak{s}})_)$ type are, up to their normalizing factors, parity transformed of each other. As can be easily checked from eq. (2.14) the operators that switch parity of bilinear quark operators $\mathfrak{s} = \bar{q}_i q_j$ and $\mathfrak{p} = \bar{q}_i \gamma_5 q_j$ are the generators $\mathbb{I}_L$ and $\mathbb{I}_R$ of the transformations $U(1)_L$ and $U(1)_R$ such that, for each matrix M, the corresponding pair of quadruplets $(\mathfrak{s}^0, \vec{\mathfrak{p}})$ and $(\mathfrak{p}^0, \vec{\mathfrak{s}})$ transform by

$$\boxed{\mathbb{I}_L.(\mathfrak{s}^0, \vec{\mathfrak{p}}) = -(\mathfrak{p}^0, \vec{\mathfrak{s}}), \quad \mathbb{I}_L.(\mathfrak{p}^0, \vec{\mathfrak{s}}) = -(\mathfrak{s}^0, \vec{\mathfrak{p}}), \quad \mathbb{I}_R.(\mathfrak{s}^0, \vec{\mathfrak{p}}) = +(\mathfrak{p}^0, \vec{\mathfrak{s}}), \quad \mathbb{I}_R.(\mathfrak{p}^0, \vec{\mathfrak{s}}) = +(\mathfrak{s}^0, \vec{\mathfrak{p}})} \tag{2.22}$$

In the following, we shall note generically Δ_i the quadruplets of the type $(\mathfrak{s}^0, \vec{\mathfrak{p}})$, and $\hat{\Delta}_i$ the ones of the type $(\mathfrak{p}^0, \vec{\mathfrak{s}})$. For 2 generations (see section 4.1), the index i can take 4 values and spans the set $\{X, H, \Omega, \Xi\}$. All $\Delta_i, \hat{\Delta}_i$ are expressed in terms of bilinear quark operators.

2.2 Generalities on Yukawa couplings

Yukawa couplings are, in the GSW model, $SU(2)_L$ invariant couplings between quarks and the Higgs doublet H, tailored to give masses to the quarks by the spontaneous breaking of the gauge symmetry. In there, H is supposed to be unique. By coupling to left-handed flavor doublets and to right-handed flavor singlets it gives masses to the quarks with charge $-1/3$ and the same procedure operated with H replaced by $i\tau^2 H^*$ gives masses to the quarks with charge $+2/3$.

We saw in subsection 2.1.3 that, as soon as one considers the possibility that the Higgs fields are bilinear quark-antiquark operators, $2N^2$ complex Higgs doublets become available. There is no reason to discard any of them such that any extension of the Standard Model with composite Higgses should include $2N^2$ complex Higgs doublets. This is what we shall do here.

The Yukawa Lagrangian that we consider for N generations is

$$\boxed{\mathcal{L}_{Yuk} = \sum_{i=1}^{N^2} -\frac{1}{2}\delta_i\big(\Delta_i^\dagger[\Delta_i] + h.c.\big) - \frac{1}{2}\delta_{\hat{i}\hat{i}}\big(\Delta_i^\dagger[\hat{\Delta}_i] + h.c.\big) - \frac{1}{2}\kappa_{\hat{i}i}\big(\hat{\Delta}_i^\dagger[\Delta_i] + h.c.\big) - \frac{1}{2}\big(\hat{\delta}_i\hat{\Delta}_i^\dagger[\hat{\Delta}_i] + h.c.\big)} \tag{2.23}$$

where the notations are the following. As we already mentioned, the $2N^2$ quadruplets that we consider are split into N^2 pairs $(\Delta_i, \hat{\Delta}_i)$ in which Δ_i and $\hat{\Delta}_i$ are parity transformed of each other. The index i in (2.23) spans this set of pairs (in the case of 1 generation (see (3.5) in section 3.2), $i = 1$ and one deals with only one pair of parity transformed Higgs quadruplets; for 2 generations i goes from 1 to 4 since one has 4 pairs of parity transformed quadruplets (see section 4.1)). Moreover, according to the notation that we introduced in (1.3) each of them can be expressed either in terms of bilinear quark operators, in which case we write it Δ_i or $\hat{\Delta}_i$, or in terms of bosonic fields (Higgs bosons, mesons ...), in which case we write it $[\Delta_i]$ or $[\hat{\Delta}_i]$.

With respect to the most general Yukawa Lagrangian, the choice (2.23) drastically reduces the number of Yukawa couplings down to $4N^2$. It has the following properties:

* it is the simplest and most straightforward generalization of the case of 1 generation. For 1 generation, as we shall see in chapter 3, (2.23) for $N = 1$ describes the most general $SU(2)_L$ invariant couplings (3.3) between the 2 quarks u, d and the two parity transformed Higgs quadruplets X and $\hat{X}$ (eqs.(3.1) and (3.2);

* it is diagonal in the index i, which means that it is a sum over pairs $(\Delta_i, \hat{\Delta}_i)$ of parity transformed doublets. It accordingly discards all crossed couplings $\Delta_i^\dagger[\Delta_j], \Delta_i^\dagger[\hat{\Delta}_j], \hat{\Delta}_i^\dagger[\hat{\Delta}_j], \hat{\Delta}_i^\dagger[\hat{\Delta}_j]$ with $i \neq j$, which forbids tree level FCNC's.

Since it involves couplings between quadruplets of opposite parity, it a priori allows classical transitions between scalars and pseudoscalars. Requesting that they vanish will provide constraints on the couplings.

By embedding the $SU(2)_L$ group into the chiral group, we have in particular established a connection between the spontaneous breaking of the two symmetries, which, due to the composite nature of the Higgs bosons, become "dynamical" because triggered by quark condensation. Now, Yukawa couplings involve couplings of the Higgs bosons to quark pairs for example $-\delta_X \left(\frac{v_X}{\sqrt{2}\mu_X^3} \frac{\bar u u+\bar d d}{\sqrt{2}}\right) X^0$, where $< X^0 >= \frac{v_X}{\sqrt{2}}, \mu_X^3 = \frac{<\bar u u+\bar d d>}{\sqrt{2}}$, such that quark condensation $< \bar u u + \bar d d > \neq 0$ triggers a linear term in the Higgs boson X^0. The first derivative of such a Yukawa coupling with respect to X^0 yields, thus, a non-vanishing constant, like does the first derivative of the term in the scalar potential quadratic in X^0 when setting X^0 to its bosonic VEV (see subsection 2.3.1). So, unlike in the genuine GSW model, Yukawa couplings are no longer passive in the construction of the vacuum, and shift its minimum with respect to what would be obtained from the sole consideration of the scalar potential. This is why, in relation to the twofold nature of the quadruplets, it is convenient to also consider, for further minimization of an "effective potential" built from the genuine one and Yukawa couplings, a bosonised form of $\mathcal{L}_{Yuk}$ which writes

$$\mathcal{L}_{Yuk}^{bos} = \sum_{i=1}^{N^2} -\delta_i[\Delta_i]^\dagger[\Delta_i] - \frac{1}{2}(\delta_{i\hat i}+\kappa_{\hat i i})\Big([\Delta_i]^\dagger[\hat\Delta_i] + [\hat\Delta_i]^\dagger[\Delta_i]\Big) - \hat\delta_i[\hat\Delta_i]^\dagger[\hat\Delta_i]. \tag{2.24}$$

In the present approach, Yukawa couplings have three roles:

* they give masses to fermions;

* they give "soft" masses to scalar and pseudoscalar mesons; exceptions are the three Goldstones of the broken $SU(2)_L$ should not become massive, which provides additional constraints;

* they modify the scalar potential and shift accordingly the position of its minima.

The bosonised form(2.24) of the Yukawa couplings can be further simplified by the requirement that no transition occurs between scalars and pseudoscalars. $[\Delta_i]^\dagger[\hat\Delta_i] = [\Delta^0][\hat\Delta^0] - [\vec\Delta].[\vec{\hat\Delta}] + i([\Delta^1][\hat\Delta^2]-[\Delta^2][\hat\Delta^1]) + [\Delta^0][\hat\Delta^3] - [\Delta^3][\hat\Delta^0]$ and $[\hat\Delta_i]^\dagger[\Delta_i] = -[\Delta^0][\hat\Delta^0] + [\vec\Delta].[\vec{\hat\Delta}] + i([\Delta^1][\hat\Delta^2]-[\Delta^2][\hat\Delta^1]) + [\Delta^0][\hat\Delta^3] - [\Delta^3][\hat\Delta^0]$ such that the middle terms in (2.24) write $-(\delta_{i\hat i}+\kappa_{\hat i i})\Big(i([\Delta^1][\hat\Delta^2]-[\Delta^2][\hat\Delta^1]) + [\Delta^0][\hat\Delta^3] - [\Delta^3][\hat\Delta^0]\Big)$. $[\Delta^{1,2,3}], [\hat\Delta^0]$ are pseudoscalars while $[\hat\Delta^{1,2,3}], [\Delta^0]$ are scalars such that the first two terms are charged scalar-pseudoscalar crossed terms. We a priori declare them unwanted at the classical level, which requires

$$\boxed{\delta_{i\hat i}+\kappa_{\hat i i} = 0} \qquad \Rightarrow \qquad \boxed{\mathcal{L}_{Yuk}^{bos} = \sum_{i=1}^{N^2} -\delta_i[\Delta_i]^\dagger[\Delta_i] - \hat\delta_i[\hat\Delta_i]^\dagger[\hat\Delta_i]} \tag{2.25}$$

2.3 Generalities on the Higgs potential

2.3.1 The scalar potential V

There, too, we shall use the simplest possible extension of the scalar potential of the GSW model to the case of $2N^2$ Higgs multiplets

$$\boxed{V = \sum_{i=1}^{N^2} -\frac{m_H^2}{2}\Big([\Delta_i]^\dagger[\Delta_i] + [\hat\Delta_i]^\dagger[\hat\Delta_i]\Big) + \frac{\lambda_H}{4}\Big(([\Delta_i]^\dagger[\Delta_i])^2 + ([\hat\Delta_i]^\dagger[\hat\Delta_i])^2\Big)} \tag{2.26}$$

(2.26) is written as a sum over the N^2 pairs $(\Delta_i, \hat\Delta_i)$ of Higgs multiplets. It only depends on 2 parameters m_H^2 and λ_H.

In there, all Higgs multiplets are written in their bosonic form $[\Delta] = ([\Delta]^0, [\Delta]^3, [\Delta]^+, [\Delta]^-)$, this is why we have used the notation with square brackets [ ].

(2.26) is not only invariant by $SU(2)_L$, but also, since all Higgs quadruplets are also complex doublets of $SU(2)_R$ (see (2.20)), by $SU(2)_R$. It is thus invariant by the chiral $SU(2)_L \times SU(2)_R$. Would the normalization factors be the same for all multiplets, V would be invariant by the larger chiral group $U(2N)_L \times U(2N)_R$. This is why we choose from the beginning to only make V depend on 2 coupling constants λ_H and m_H^2: the underlying $U(2N)_R \times U(2N)_L$ symmetry gets spontaneously broken down to $SU(2)_L \times SU(2)_R$ by the non equality of the bosonic VEV's which break the gauge symmetry, $v_i \neq v_j \neq \hat{v}_k \neq \hat{v}_l, i, j, k, l \in [1, N^2]$ and, likewise, by the non equality of the fermionic VEV's ($< \bar{q}q >$) $\mu_i^3 \neq \mu_j^3 \neq \hat{\mu}_k^3 \neq \hat{\mu}_l^3$ (the non-vanishing of the fermionic VEV's is usually attributed "chiral symmetry breaking"). Since the gauge group has been embedded in the chiral group, the situation is of course very intricate.

Additional remarks concerning the Higgs potential can be found in section 3.5 for 1 generation of quarks.

2.3.2 The effective scalar potential V_{eff}

V_{eff} is defined to be the difference between the genuine potential (2.26) and the bosonised Yukawa Lagrangian (2.25)

$$\boxed{V_{eff} = V - \mathcal{L}_{Yuk}^{bos}} \tag{2.27}$$

Like V, it is invariant by the chiral group $SU(2)_L \times SU(2)_R$.

Each quadruplet Δ_i and $\hat{\Delta}_i$ includes a priori one Higgs boson, that is one scalar with a non-vanishing VEV: $\Delta_i^0 \in \Delta_i$ and $\hat{\Delta}_i^3 \in \hat{\Delta}_i$. It is usual matter, like in the GSW model, to find that the minimum of the *genuine* potential V given in (2.26) occurs at $< \Delta_i^0 >^2 = \dfrac{m_H^2}{\lambda_H} = < \hat{\Delta}_i^3 >^2$ for all quadruplets. Since we write the bosonic VEV's $< \Delta_i^0 > = \frac{v_i}{\sqrt{2}}$ and $< \hat{\Delta}_i^3 > = \frac{\hat{v}_i}{\sqrt{2}}$, it is then natural to define

$$\boxed{v_0^2 = \frac{2m_H^2}{\lambda_H}} \tag{2.28}$$

For later use we shall also define the parameter δ such that

$$\boxed{\lambda_H = \frac{4\delta}{v_0^2} \Rightarrow m_H^2 = 2\delta} \tag{2.29}$$

Because, as we saw, Yukawa couplings are non longer passive in the definition of the (spontaneously broken) vacuum of the theory, the latter is now defined by the minimum of V_{eff} with respect to the $2N^2$ Higgs bosons. It is obtained from the one from V with the simple shift $\frac{m_H^2}{2} \to \frac{m_H^2}{2} - \delta_i$ for a Higgs boson Δ_i^0 and $\frac{m_H^2}{2} \to \frac{m_H^2}{2} - \hat{\delta}_i$ for a Higgs boson Δ_i^3. The equations tuning to zero the corresponding first derivatives are thus of the type

$$m_H^2 - 2\delta_i = \lambda_H < \Delta_i^0 >^2, \quad m_H^2 - 2\hat{\delta}_i = \lambda_H < \hat{\Delta}_i^3 >^2, \tag{2.30}$$

that is, using (2.29) and (2.28)

$$\boxed{\delta_i = \delta(1 - b_i), \quad \hat{\delta}_i = \delta(1 - \hat{b}_i)} \tag{2.31}$$

in which we have defined

$$\boxed{b_i = \frac{v_i^2}{v_0^2}, \quad \hat{b}_i = \frac{\hat{v}_i^2}{v_0^2}} \tag{2.32}$$

2.3.3 Goldstones and pseudo-Goldstones

In the absence of the Yukawa couplings, the spontaneous breaking of $SU(2)_L$ would produce 3 Goldstone bosons inside each Higgs quadruplet.

Physically, only three of them can become the longitudinal $W_\parallel$'s, and the rest of them is therefore doomed to get "soft" masses from the Yukawa couplings, becoming what is commonly called "pseudo-Goldstone bosons". So, the spectrum of the theory is expected to be composed, after symmetry breaking, of $2N^2$ Higgs bosons and $6N^2-3$ pseudo-Goldstone bosons.

The $mass^2$ of the pseudo-Goldstones must be calculated from the second derivative of $V-\mathcal{L}_{Yuk}$. The result, as we show below on the same simple example of the $X^\pm$ bosons, turns out to be the same as if it were calculated from the sole term $\delta_X X^\dagger X$ of $(-)$ the bosonised Yukawa Lagrangian (2.25).

$-\mathcal{L}_{Yuk}+V$ involve the couplings

$$\begin{aligned} &-\frac{1}{2}\delta_X\left(X^+\left(\frac{v_X}{\sqrt{2}\mu_X^3}\frac{1}{\sqrt{2}}2\bar d\gamma_5 u\right)+X^-\left(\frac{v_X}{\sqrt{2}\mu_X^3}\frac{1}{\sqrt{2}}2\bar u\gamma_5 d\right)+\ldots\right)\\ &-\frac{m_H^2}{2}(-X^+X^-+\ldots)+\frac{\lambda_H}{4}\Big(-X^+X^- -(X^3)^2+(X^0)^2\Big)^2, \end{aligned} \tag{2.33}$$

in which we have only written terms which are relevant for the first and second derivatives when $<X^0>=\frac{v_X}{\sqrt{2}}$. This yields, using (2.29), (2.31) and (2.32)

$$\frac{\partial^2(-\mathcal{L}_{Yuk}+V)}{\partial X^+\partial X^-}\overset{<X^0>=v_X/\sqrt{2}}{=}\frac{m_H^2}{2}-\frac{\lambda_H}{4}v_X^2=\delta(1-b_X)=\delta_X, \tag{2.34}$$

which is the result announced above.

This could be naively interpreted by saying that the $SU(2)_L$ Goldstones that occur in V after symmetry breaking get, afterwards, their soft masses from the bosonised Yukawa couplings. This maybe right in practice but conceptually erroneous. To see this it is enough to minimize the effective potential $V_{eff}=V-\mathcal{L}^{bos}_{Yuk}$: at the appropriate minimum, all $6N^2$ Goldstones are true massless Goldstones.

The consequence is that, if we call $\hat H$ the quadruplet that contains the 3 Goldstones doomed to become the 3 longitudinal gauge bosons, its Yukawa couplings should satisfy

$$\boxed{\hat\delta_H=0} \tag{2.35}$$

which adds to the N^2 constraints in the l.h.s. of (2.25). Because of (2.35), the effective potential V_{eff} (2.25) for the Higgs multiplet $\hat H$ is identical to the genuine one V. It has accordingly its minimum at

$$\boxed{\hat v_H=v_0} \tag{2.36}$$

where v_0 has been defined in (2.28).

2.4 The masses of the Higgs bosons

Let us call generically $Higgs$ a Higgs boson, which has a VEV $\frac{v_{Higgs}}{\sqrt{2}}$. The corresponding quantum Higgs field h is defined by the shift $Higgs=\frac{v_{Higgs}}{\sqrt{2}}+h$ and its squared mass m_h^2 is given by

$$m_h^2=\frac{1}{2}\frac{\partial^2 V_{eff}}{(\partial\Delta_0)^2}\Big|_{<\Delta_0>=\frac{v_{Higgs}}{\sqrt{2}}}. \tag{2.37}$$

One gets accordingly for the "quasi-standard" Higgs boson $\hat H^3$

$$\boxed{m^2_{\hat H^3}=2\delta=m_H^2} \tag{2.38}$$

and for the others

$$\boxed{m^2_{Higgs}=2\delta\, b_{Higgs}=m^2_{\hat H^3}\, b_{Higgs}} \tag{2.39}$$

where, according to (2.32), we have defined $b_{Higgs}=(v_{Higgs}/\hat v_H)^2$ and we have used (2.36).

This shows in particular that the set of parameters made of δ and the b's are fundamental concerning the spectrum of the theory.

Chapter 3

The case of 1 generation

This toy model is already a 2-Higgs doublet model. We show how all parameters can be determined from pion physics, the W mass and fermionic constraints, and how the leptonic decays of charged pions are suitably described. The "quasi-standard" Higgs boson has mass $\sqrt{2}m_\pi$, and the second Higgs boson is very light ($69\,KeV$).

3.1 The 2 Higgs quadruplets

This case only involves 2 parity-transformed Higgs multiplets [1], 4 VEV's, 3 pions, 2 Higgs bosons and 3 Goldstone bosons. The number of independent Yukawa couplings reduces to 2 and the equations are easy to solve, also simplified by the absence of mixing.

The 2 Higgs multiplets are the following

$$\begin{aligned} X &= \frac{v_X}{\sqrt{2}\,\mu_X^3}\frac{1}{\sqrt{2}}\,\bar\psi\left(\begin{pmatrix} 1 & \\ & 1\end{pmatrix}, \gamma^5\begin{pmatrix} 1 & \\ & -1\end{pmatrix}, 2\gamma^5\begin{pmatrix} & 1 \\ 0 & \end{pmatrix}, 2\gamma^5\begin{pmatrix} & 0 \\ 1 & \end{pmatrix}\right)\psi \\ &= (X^0, X^3, X^+, X^-), \quad with \quad \mu_X^3 = \frac{< \bar u u + \bar d d >}{\sqrt{2}}, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \hat X &= \frac{\hat v_X}{\sqrt{2}\,\hat\mu_X^3}\frac{1}{\sqrt{2}}\,\bar\psi\left(\gamma_5\begin{pmatrix} 1 & \\ & 1\end{pmatrix}, \begin{pmatrix} 1 & \\ & -1\end{pmatrix}, 2\begin{pmatrix} & 1 \\ 0 & \end{pmatrix}, 2\begin{pmatrix} & 0 \\ 1 & \end{pmatrix}\right)\psi \\ &= (\hat X^0, \hat X^3, \hat X^+, \hat X^-), \quad with \quad \hat\mu_X^3 = \frac{< \bar u u - \bar d d >}{\sqrt{2}}, \end{aligned} \tag{3.2}$$

in which $\psi = \begin{pmatrix} u \\ d \end{pmatrix}$. The 2 Higgs bosons are X^0 and $\hat X^3$. Since X contains a triplet of pseudoscalars which we wish to identify with the three pions, $\hat X$ is the quadruplet doomed to incorporate the 3 Goldstone bosons of the broken $SU(2)_L$ that become the 3 longitudinal gauge bosons. This is why we shall call $\hat X^3$ the "quasi-standard" Higgs boson, and we shall keep this terminology in the rest of the paper. We see that, while the two charged Goldstones are the two scalars $[\hat X]^+, [\hat X]^-$, the neutral one is the pseudoscalar $[\hat X]^0$. Parity is thus, there, obviously broken.

[1]Among all the 2-Higgs doublets models that have been investigated [17], none involve parity-transformed multiplets. This is probably due to the fact that the laws of transformations (2.6) and (2.18) were never written before.

3.2 The Yukawa couplings

We start writing them in the "standard" form, in which, as usual, a given complex Higgs doublet H must be associated to its complex conjugate $i\tau^2 H^*$ to provide masses for both d-type and u-type quarks. This means that we use $[X], [\hat{X}]$ together with $i\tau^2[X]^*, i\tau^2[\hat{X}]^*$ and write the Yukawa couplings as

$$\begin{aligned}\mathcal{L}_{Yuk} = &+ \rho_d \left(\overline{u_L}\, \overline{d_L} \right) [X]\, d_R - \rho_u \left(\overline{u_L}\, \overline{d_L} \right) (i\tau^2 [X]^*)\, u_R \\ &+ \lambda_d \left(\overline{u_L}\, \overline{d_L} \right) [\hat{X}]\, d_R + \lambda_u \left(\overline{u_L}\, \overline{d_L} \right) (i\tau^2 [\hat{X}]^*)\, u_R \\ &+ h.c.,\end{aligned} \tag{3.3}$$

in which the Higgs multiplets $[X]$ and $[\hat{X}]$ are written as $SU(2)_L$ doublets, for example $[X] = \begin{pmatrix} [X^1] - i[X^2] \\ -[X^0] - [X^3] \end{pmatrix}$ and, therefore, $i\tau^2[X]^* = \begin{pmatrix} [X^0] - [X^3] \\ -[X^1] - i[X^2] \end{pmatrix}$.

The expression (3.3) represents the most general Yukawa couplings between 2 quarks and 2 Higgs doublets. After suitably grouping terms, it gives, explicitly [2],

$$\mathcal{L}_{Yuk} = -\frac{1}{2}\delta_X (X^\dagger [X] + h.c.) - \frac{1}{2}\delta_{X\hat{X}} \big(X^\dagger [\hat{X}] + h.c.\big) - \frac{1}{2}\kappa_{\hat{X}X} \big(\hat{X}^\dagger [X] + h.c.\big) - \frac{1}{2}\hat{\delta}_X (\hat{X}^\dagger [\hat{X}] + h.c.). \tag{3.5}$$

In (3.3) and (3.5) the signs have been set such that for positive $<\mathfrak{s}^0>$ and $<\mathfrak{s}^3>$, the fermion masses are positive for positive $\rho_{u,d}$ and $\lambda_{u,d}$ (given that a fermion mass term is of the form $-m\bar{\psi}\psi$) and, in (3.5), we have introduced the parameters with dimension $[mass]^2$

$$\begin{aligned}\delta_X &= \frac{\rho_u + \rho_d}{\sqrt{2}} \frac{\mu_X^3}{v_X/\sqrt{2}}, \\ \kappa_{\hat{X}X} &= \frac{\rho_u - \rho_d}{\sqrt{2}} \frac{\hat{\mu}_X^3}{\hat{v}_X/\sqrt{2}}, \\ \delta_{X\hat{X}} &= \frac{\lambda_u + \lambda_d}{\sqrt{2}} \frac{\mu_X^3}{v_X/\sqrt{2}}, \\ \hat{\delta}_X &= \frac{\lambda_u - \lambda_d}{\sqrt{2}} \frac{\hat{\mu}_X^3}{\hat{v}_X/\sqrt{2}}.\end{aligned} \tag{3.6}$$

It is then trivial matter, by replacing in (3.5), the bilinear quark operators by the corresponding bosonic fields, to get the bosonised form of the Yukawa couplings

$$\mathcal{L}^{bos}_{Yuk} = -\delta_X [X]^\dagger [X] - \frac{1}{2} \underbrace{(\delta_{X\hat{X}} + \kappa_{\hat{X}X})}_{0} \big([X]^\dagger [\hat{X}] + [\hat{X}]^\dagger [X]\big) - \underbrace{\hat{\delta}_X}_{0} [\hat{X}]^\dagger [\hat{X}], \tag{3.7}$$

in which the second and third term vanish from the general constraints (2.25) (2.35) that we have established $\delta_{X\hat{X}} + \kappa_{\hat{X}X} = 0$, $\hat{\delta}_X = 0$ [3]. This 2-Higgs doublet model has thus only 2 Yukawa couplings δ_X and $\delta_{X\hat{X}}$.

[2] As can be easily verified

$$\begin{aligned}\frac{1}{2}(X^\dagger [X] + h.c.) &= X^0[X^0] - X^3[X^3] - \frac{1}{2}(X^+[X^-] + X^-[X^+]), \\ \frac{1}{2}(X^\dagger [\hat{X}] + h.c.) &= X^0[\hat{X}^3] - X^3[\hat{X}^0] + \frac{1}{2}(X^-[\hat{X}^+] - X^+[\hat{X}^-]), \\ \frac{1}{2}(\hat{X}^\dagger [X] + h.c.) &= -\hat{X}^0[X^3] + \hat{X}^3[X^0] - \frac{1}{2}(\hat{X}^-[X^+] - \hat{X}^+[X^-]), \\ \frac{1}{2}(\hat{X}^\dagger [\hat{X}] + h.c.) &= -\hat{X}^0[\hat{X}^0] + \hat{X}^3[\hat{X}^3] + \frac{1}{2}(\hat{X}^+[\hat{X}^-] + \hat{X}^-[\hat{X}^+]),\end{aligned} \tag{3.4}$$

in which the scalar or pseudoscalar nature of the different fields has been taken into account for hermitian conjugation and all VEV's have been supposed to be real.

[3] In this case, $[\hat{X}]$ is the quadruplet that contains the three $SU(2)_L$ Goldstones such that (2.35) writes $\hat{\delta}_X = 0$.

As a side remark, notice that a term proportional to $[X]^\dagger[\hat X]+[\hat X]^\dagger[X]$, which can be potentially present in $\mathcal{L}_{Yuk}^{bos}$, includes terms $[X^0][\hat X^3] - [\hat X^0][X^3]$, the last being crossed couplings between the 2 pseudoscalars $[\bar u\gamma_5 u + \bar d\gamma_5 d] \propto \eta$ and $[\bar u\gamma_5 u - \bar d\gamma_5 d] \propto \pi^0$. Since $\hat\delta_X = 0$ for the preservation of the 3 $SU(2)_L$ Goldstones, this binary system would have a matrix for its squared masses proportional to $\left(\begin{array}{cc} \delta_X & \frac{\delta_{X\hat X}+\kappa_{\hat X X}}{2} \\ \frac{\delta_{X\hat X}+\kappa_{\hat X X}}{2} & 0 \end{array}\right)$ and would exhibit a tachyonic state without the condition $\delta_{X\hat X} + \kappa_{\hat X X} = 0$.

3.3 Normalizing the pions

As was already written in (1.2) and (1.3) and since there is no mixing, PCAC [4]

$$\partial_\mu(\bar u\gamma_5\gamma^\mu d) = i(m_u+m_d)\bar u\gamma_5 d = \sqrt{2} f_\pi m_\pi^2 \pi^+ \tag{3.8}$$

and the GMOR relation

$$(m_u+m_d) < \bar u u + \bar d d >= 2 f_\pi^2 m_\pi^2 \tag{3.9}$$

entail that

$$[X^\pm] = \mp i \frac{v_X}{f_\pi}\pi^\pm. \tag{3.10}$$

The kinetic terms are written in the standard way

$$\mathcal{L}_{kin} = (D_\mu[X])^\dagger D^\mu[X] + (D_\mu[\hat X])^\dagger D^\mu[\hat X], \tag{3.11}$$

where D_μ is the covariant derivative with respect to the $SU(2)_L$ group. Since $[X]^\dagger[X] \ni -[X^+][X^-]$ the charged pions will be normalized in a standard way if

$$v_X = f_\pi, \tag{3.12}$$

and, according to subsection 2.3.3, the masses of these pseudo-Goldstones will correspond to the pion mass if

$$\delta_X = m_\pi^2. \tag{3.13}$$

3.4 The mass of the gauge bosons

The W gauge bosons get their masses from the VEV's of the 2 Higgs bosons $[\hat X^3]$ and $[X^0]$; from the kinetic terms (3.11) one gets

$$m_W^2 = \frac{g^2}{2}\big(< [X^0] >^2 + < [\hat X^3] >^2 \big) = g^2 \frac{v_X^2 + \hat v_X^2}{4}, \tag{3.14}$$

in which g is the $SU(2)_L$ coupling constant (in (3.15) below G_F is the Fermi constant)

$$\frac{g^2}{8m_W^2} = \frac{G_F}{\sqrt{2}} \Rightarrow g \approx .61. \tag{3.15}$$

Because v_X given by (3.12) is $\ll m_W$,

$$\hat v_X \approx \frac{2m_W}{g}. \tag{3.16}$$

The hierarchy of the two bosonic VEV's for 1 generation is thus very large

$$\frac{< [\hat X^3] >}{< [X^0] >} = \frac{\hat v_X/\sqrt{2}}{v_X/\sqrt{2}} \approx \frac{2m_W/g}{f_\pi} \approx 2858. \tag{3.17}$$

[4] We use PCAC with a factor $\sqrt{2}$ in the r.h.s. of (5.37), such that the corresponding value of f_π is $f_\pi \approx 93\, MeV$.

3.5 The scalar potential and the masses of the Higgs bosons

• The genuine potential V we take as

$$V([X],[\hat X]) = -\frac{m_H^2}{2}\left([X]^\dagger[X]+[\hat X]^\dagger[\hat X]\right)+\frac{\lambda_H}{4}\Big(([X]^\dagger[X])^2+([\hat X]^\dagger[\hat X])^2\Big). \tag{3.18}$$

With respect to the most general potential for two Higgs doublets the following terms have accordingly been discarded:

* $(m^2 X^\dagger\hat X + h.c)$, with $m\in\mathbb{C}$ would mediate in particular transitions between scalars and pseudoscalars that should not occur classically;

* $\lambda_4(X^\dagger X)(X^\dagger\hat X)+h.c.$, $\lambda_5(\hat X^\dagger\hat X)(X^\dagger\hat X)+h.c.$ with $\lambda_4,\lambda_5\in\mathbb{C}$ would also mediate unwanted classical transitions between scalars and pseudoscalars;

* $\lambda_3(X^\dagger\hat X)^2+h.c.$ with $\lambda_3\in\mathbb{C}$ would in particular contribute to the mass of the neutral pion and not to that of the charged pions. Such a classical $\pi^+-\pi^0$ mass difference which is not electromagnetic nor due to $m_u\neq m_d$ is unwelcome;

* $\lambda_1(X^\dagger X)(\hat X^\dagger\hat X)$, $\lambda_2(X^\dagger\hat X)(\hat X^\dagger X)$, with $\lambda_1,\lambda_2\in\mathbb{R}$ would also spoil the Goldstone nature of the pions and η, the first because of terms proportional to $<[\hat X^3]>^2\ \vec\pi^2$ and $<[X^0]>^2\ \eta^2$, the second because of terms proportional to $<[X^0]>^2\ \eta^2, <[\hat X^3]>^2\ {\pi^0}^2$ and $<[X^0]><[\hat X^3]>\pi^0\eta$.

• Owing to (2.27), (2.25) and (2.35) the effective potential for the Higgs bosons is

$$\begin{aligned}V_{eff}([X],[\hat X]) &= V([X],[\hat X]) - \mathcal{L}_{Yuk}^{bos}([X],[\hat X])\\ &= -\frac{m_H^2}{2}\left([X]^\dagger[X]+[\hat X]^\dagger[\hat X]\right)+\frac{\lambda_H}{4}\Big(([X]^\dagger[X])^2+([\hat X]^\dagger[\hat X])^2\Big)+\delta_X[X]^\dagger[X].\end{aligned} \tag{3.19}$$

• It minimization with respect to $[\hat X^3]$ yields as expected

$$<[\hat X^3]>^2=\frac{v_0^2}{2}\Leftrightarrow \hat v_X = v_0, \tag{3.20}$$

which entails in particular

$$b_X \stackrel{(2.32)}{\equiv}\left(\frac{v_X}{v_0}\right)^2=\left(\frac{v_X}{\hat v_X}\right)^2=\left(\frac{f_\pi}{2m_W/g}\right)^2\approx\frac{1}{2858^2}. \tag{3.21}$$

Then, by (2.31)

$$\delta=\frac{\delta_X}{1-b_X}\approx\delta_X=m_\pi^2. \tag{3.22}$$

The second derivative of V_{eff} yields the mass squared of the corresponding Higgs boson $\hat x^3$ defined by $[\hat X^3]=\frac{\hat v_X}{\sqrt2}+\hat x^3$

$$m_{\hat x^3}^2=2\delta\approx 2\delta_X=2m_\pi^2\approx(194\,MeV)^2. \tag{3.23}$$

• The minimization of V_{eff} with respect to $[X^0]$ yields at expected

$$<[X^0]>^2=b_X\frac{v_0^2}{2}, \tag{3.24}$$

and the mass of the second Higgs boson x^0 defined by $[X^0]=\frac{v_X}{\sqrt2}+x^0$ is

$$m_{x^0}^2=b_X m_{\hat x^3}^2\approx(68\,KeV)^2. \tag{3.25}$$

As a side product of (3.20) one gets from (2.29)

$$\lambda_H=\frac{4\delta}{\hat v_X^2}\approx g^2\frac{m_\pi^2}{m_W^2}\ll 1. \tag{3.26}$$

Several remarks are in order.
* the ration of the masses of the 2 Higgs bosons is the same as the ratio of their VEV's $\frac{m_{\hat{x}^3}}{m_{x^0}} = \frac{\hat{v}_X/\sqrt{2}}{v_X/\sqrt{2}} = \frac{2m_W/g}{f_\pi}$;
* a very light scalar x^0 appears, with a mass of a few ten KeV;
* the "quasi-standard" Higgs boson is still very light, its mass being $\simeq m_\pi$; this shows that the "Higgs mass" is not likely to be set by m_W but rather by the mass of pseudoscalar mesons, themselves in relation with the masses of quarks. We shall confirm this fact in the case of 2 generations;
* that the Higgs mass may increase when more generations are added in indeed good sign, but then (3.26) may cause problems if m_π is replaced my m_W/g, since the quartic coupling of the Higgs potential becomes then larger than 1, such that perturbative techniques can no longer be used.

3.6 Quark masses determine the last two parameters

$\mu_X^3 \equiv \frac{<\bar{u}u+\bar{d}d>}{\sqrt{2}}$ is related to the masses of u and d quarks and to m_π^2 by the GMOR relation (3.9) such that, combined with (3.12), the first equation of (3.6) yields $\rho_u + \rho_d = \frac{m_u+m_d}{\sqrt{2}f_\pi}$.

Because of the condition (2.25), the second and third equations of (3.6) yield $\rho_u - \rho_d = -(\lambda_u+\lambda_d)\frac{\hat{v}_X}{v_X}\frac{\mu_X^3}{\hat{\mu}_X^3}$, in which $\frac{\hat{v}_X}{v_X} = \frac{2m_W/g}{f_\pi} = \sqrt{b_X}$ has been determined in (3.21), and μ_X^3 is known.

From the last of eqs. (3.6) and the condition $\hat{\delta}_X = 0$ (see footnote 3) one knows that $\lambda_u = \lambda_d = \lambda_{u,d}$; then the third equation of this same set yields $\delta_{X\hat{X}} = \lambda_{u,d}\,\frac{2\mu_X^3}{v_X}$.

Last, 2 more relations are given by the u and d quark masses, which spring from the VEV's of both Higgs bosons according to

$$m_u = \rho_u < [X^0] > +\lambda_u < [\hat{X}^3] >= \frac{v_X\rho_u + \hat{v}_X\lambda_u}{\sqrt{2}}, \quad m_d = \rho_d < [X^0] > +\lambda_d < [\hat{X}^3] >= \frac{v_X\rho_d + \hat{v}_X\lambda_d}{\sqrt{2}}, \tag{3.27}$$

which entail, using the results just obtained above

$$\begin{aligned} m_u + m_d &= (\rho_u+\rho_d)\frac{v_X}{\sqrt{2}} + (\lambda_u+\lambda_d)\frac{\hat{v}_X}{\sqrt{2}} = \frac{m_u+m_d}{2} + 2\lambda_{u,d}\frac{\hat{v}_X}{\sqrt{2}} \Rightarrow \lambda_{u,d} = \frac{m_u+m_d}{4\hat{v}_X/\sqrt{2}},\\ m_u - m_d &= (\rho_u-\rho_d)\frac{v_X}{\sqrt{2}} + \underbrace{(\lambda_u-\lambda_d)}_{0}\frac{\hat{v}_X}{\sqrt{2}} = -2\lambda_{u,d}\frac{\hat{v}_X}{\sqrt{2}}\frac{\mu_X^3}{\hat{\mu}_X^3} = -\frac{m_u+m_d}{2}\frac{\mu_X^3}{\hat{\mu}_X^3}. \end{aligned} \tag{3.28}$$

The last equation in (3.28) yields

$$\hat{r}_X \equiv \frac{\hat{\mu}_X^3}{\mu_X^3} \equiv \frac{<\bar{u}u-\bar{d}d>}{<\bar{u}u+\bar{d}d>} = -\frac{1}{2}\frac{m_u+m_d}{m_u-m_d}, \tag{3.29}$$

and the first

$$\delta_{X\hat{X}} = -\kappa_{\hat{X}X} = \frac{(m_u+m_d)\mu_X^3}{\sqrt{2}v_X\hat{v}_X} \approx \frac{f_\pi m_\pi^2}{2m_W/g}, \tag{3.30}$$

in which we have used the GMOR relation (3.9) and the values of v_X and $\hat{v}_X$ that we have obtained in (3.12) and (3.16).

3.7 Summary and comments for 1 generation

We have now determined all parameters and the masses of the 2 Higgs bosons in terms of physical quantities: the mass of the W's, the pions mass and their decay constant f_π, the $SU(2)_L$ coupling constant g and the 2 quark masses m_u and m_d. We summarize the results below.

- "bosonic" VEV's : $< X^0 > \equiv \frac{v_X}{\sqrt{2}} = \frac{f_\pi}{\sqrt{2}}, \quad < \hat{X}^3 > \equiv \frac{\hat{v}_X}{\sqrt{2}} = \frac{\sqrt{2} m_W}{g},$
- "fermionic" VEV's : $< \bar{u}u + \bar{d}d > \stackrel{(3.9)}{=} \frac{2 f_\pi^2 m_\pi^2}{m_u + m_d}, \quad \frac{< \bar{u}u - \bar{d}d >}{< \bar{u}u + \bar{d}d >} = -\frac{1}{2} \frac{m_u + m_d}{m_u - m_d},$
- Higgs bosons masses : $m_{\hat{x}^3} = \sqrt{2} m_\pi, \quad m_{x^0} = m_{\hat{x}^3} \sqrt{\frac{< X^0 >}{< \hat{X}^3 >}} = \sqrt{2} m_\pi \sqrt{\frac{f_\pi}{2 m_W / g}},$ (3.31)
- Yukawa couplings : $\delta_X = m_\pi^2, \quad \hat{\delta}_X = 0, \quad \delta_{X\hat{X}} = -\kappa_{\hat{X}X} \approx \frac{f_\pi m_\pi^2}{2 m_W / g},$
- scalar couplings : $m_H^2 = 2\delta \approx 2\delta_X = 2 m_\pi^2, \quad \lambda_H = \frac{4\delta}{\hat{v}_X^2} \approx \frac{g^2 m_\pi^2}{m_W^2} \ll 1.$

We already notice that the second equation in (3.31) points at $m_d < 0$ [5] such that

$$\frac{< \bar{u}u - \bar{d}d >}{< \bar{u}u + \bar{d}d >} = \frac{1}{2} \frac{|m_d| - m_u}{|m_d| + m_u} \approx \frac{1}{6}. \tag{3.32}$$

That $m_d < 0$ is needed will be confirmed in the case of 2 generations [6] .

The masses of the gauge bosons and of the pions have been used as inputs and are therefore suitably accounted for. The mass of the quasi-standard Higgs boson is still much too low, but the situation will improve for 2 generations, showing that this mass is connected with that of the heaviest pseudoscalar bound state. The second Higgs boson is a very light scalar. A detailed study of such light states and their couplings will be the subject of a forthcoming work [18], in which all couplings between the different fields will be specially scrutinized because, there, lies non-standard physics.

The hierarchy between the "bosonic" VEV's of the 2 Higgs bosons is very large, but will be replaced by more numerous but smaller hierarchies when 1 generation is added.

3.8 The leptonic decay $\pi^+ \to \ell^+ \nu_\ell$

It is generally believed than any tentative to build a composite "standard-like" Higgs boson with usual quarks is doomed to failure because of the factor f_π / m_W. Either the mass of the Higgs boson that one gets is $\mathcal{O}(f_\pi, m_\pi)$, or leptonic decays are off by the factor f_π / m_W because the longitudinal component of the massive W is also the pseudo-goldstone pion (see [2]).

This argumentation becomes void in our framework, as already briefly mentioned in the introduction, due to the following properties:
* we are dealing with a model with several composite Higgs multiplets;
* to every such multiplet is attached two VEV's: a bosonic one, like the $v/\sqrt{2}$ of the GSW model, and a fermionic one directly connected to $< \bar{q}q >$ quark condensates: there are now enough scales to accommodate for for both chiral and weak physics;
* low energy considerations fix the normalizations of a set of charged pseudoscalar composite states, like the ones associated to $\pi^\pm, K^\pm, \ldots$. That the corresponding mesons get no longer normalized to 1 provides the ultimate mechanism to reconcile "pion" physics and weak symmetry breaking.

[5] otherwise one gets $\frac{< \bar{u}u - \bar{d}d >}{< \bar{u}u + \bar{d}d >} \approx \frac{3}{2}$.

[6] In the GSW model, the sign of the fermion masses is irrelevant, at least at the classical level since swapping the sign involves a γ_5 transformation, which should be anomaly-free. Indeed, by a transformation $d \to e^{i\beta\gamma_5} d$, $\bar{d}d$ becomes $(\cos 2\beta\, \bar{d}d + i \sin 2\beta\, \bar{d}\gamma^5 d)$ such that, for $\beta = \pi/2$, this is equivalent to swapping the sign of m_d. One must however keep in mind that, in the present framework, this transformation must also be operated on all quark bilinears: for example, $\bar{u}\gamma^5 d$ becomes $i\bar{u}d$ and the parity of bilinears involving one d quark gets swapped. In the 1-generation case, the components of the X quadruplet (3.1) become proportional to $(\bar{u}u - \bar{d}d, \bar{u}\gamma^5 u + \bar{d}\gamma^5 d, 2i\, \bar{u}d, 2i\, \bar{d}u)$ such that, in particular, $< \bar{u}u + \bar{d}d >$ becomes $< \bar{u}u - \bar{d}d >$.

The argumentation that we develop below is independent of the relation (3.12) and goes, as we shall see, beyond the trivial case of 1 generation.

$\pi^+ \to \ell^+\nu_\ell$ decays are triggered by the crossed terms

$$\partial^\mu X^+ (ig) W_\mu^- \underbrace{T_L^+ . X^-}_{-X^0 - X^3} \tag{3.33}$$

which arises in the kinetic Lagrangian for the X quadruplet. Since X^0 gets a VEV $v_X/\sqrt{2}$, (3.33) induces the coupling

$$-ig\frac{v_X}{\sqrt{2}}\,\partial^\mu X^+ W_\mu^-. \tag{3.34}$$

Using PCAC and the GMOR relation, we have already shown (see for example in the introduction) that

$$X^+ = -i\frac{v_X}{f_\pi}\pi^+ = a_X\pi^+, \quad a_X = -i\frac{v_X}{f_\pi}. \tag{3.35}$$

From (3.35) we deduce that the matrix element $\mathcal{M}_\pi = \langle \ell\nu_\ell | \pi^+\rangle$ is given by

$$\mathcal{M}_\pi = \frac{1}{a_X}\underbrace{\langle \ell\nu_\ell | X^+\rangle}_{\mathcal{M}_X}. \tag{3.36}$$

$\mathcal{M}_X$ can be easily calculated from the genuine Lagrangian, since, in there, all fields are "normalized to 1". From (3.36) and using the unitary gauge for the W, one gets [7]

$$\mathcal{M}_X = -ig\frac{v_X}{\sqrt{2}}(ip_\mu)\frac{-ig_{\mu\nu}}{p^2 - m_W^2}(ig)\; v_\ell(k)\gamma^\mu\frac{1-\gamma_5}{2}\bar{v}_{\nu_\ell}(k'), \tag{3.37}$$

and thus, by (3.36) and at $p^2 = m_\pi^2 \ll m_W^2$

$$\mathcal{M}_\pi = ip_\mu\frac{g^2 f_\pi}{\sqrt{2}\,m_W^2}\; v_\ell(k)\gamma^\mu\frac{1-\gamma_5}{2}\bar{v}_{\nu_\ell}(k'), \tag{3.38}$$

which is the "standard" PCAC amplitude. It is controlled by the product $v_X/a_X \simeq f_\pi$ *independent of* v_X [8].

Miscellaneous remarks

• In the usual S-matrix formalism [9], $\langle 0\, |X^+(x)|\pi^+(p)\rangle = \sqrt{Z}\langle 0\, |X^+_{in}(x)|\pi^+(p)\rangle$ with $\langle 0\, |X^+_{in}(x)|\pi^+(p)\rangle = \int d^3q\; e^{-iqx}\langle 0\, |a_{in}(q)|\pi^+(p)\rangle = e^{-ipx}$, $\sqrt{Z}$ represents the amplitude for creating a 1-pion state from the vacuum with $X^+(x)$. In our case, $\sqrt{Z} \simeq v_X/f_\pi$. For 1 generation, it is equal to 1 (see (3.12), but becomes much larger for 2 generations.

• While the components of X are interacting (composite) bosonic fields entering the Lagrangian, $\pi^\pm$ are the physical asymptotic states. So as to calculate the decay rate, it is these physical states that we normalize according to

$$\langle p'\; in | p\; in\rangle = 2p^0(2\pi)^3\delta(\vec{p} - \vec{p}'), \tag{3.39}$$

such that

$$|p\rangle = \int \underbrace{\frac{d^4p'}{(2\pi)^3}\theta(p'_0)\delta(p'^2 - m^2)}_{d\mu(p')}|p'\rangle\langle p'|p\rangle. \tag{3.40}$$

The phase space measure for outgoing particles is as usual

$$d\mu(k) = \delta(k^2 - m)\; d^4k\frac{1}{(2\pi)^3}\theta(k_0). \tag{3.41}$$

and

$$d\Gamma = \frac{1}{2m_\pi}|\mathcal{M}_\pi|^2(2\pi)^4\delta^4(p - k - k')\; d\mu(k)\; d\mu(k'). \tag{3.42}$$

[7] In the two equations below, $v_\ell, \bar{v}_{\nu_\ell}$ stand for Dirac spinors and should not be confused with bosonic VEV's.
[8] such that v_X can take any value, like $v_x \simeq \hat{v}_H$ for 2 generations.
[9] See for example [19], chapt. 16.

Chapter 4

The case of 2 generations . General results

We give here basic formulæ that will be used in the rest of the paper: explicit expressions of the quadruplets, notations, kinetic terms and gauge bosons mass, masses and orthogonality relations for pseudoscalar mesons, and the master formula (1.8) linking the mixing angles θ_d and θ_u to the masses of charged pseudoscalars. We also introduce the group $U(2)^g_L \times U(2)^g_R$ that moves inside the space of quadruplets.

4.1 The 8 Higgs quadruplets

According to (2.15), we consider hereafter the 4 following quadruplets of the type $(\mathfrak{s}^0, \vec{\mathfrak{p}})$ (ψ stands now for $(u,c,d,s)^t$)

$$\boxed{\begin{aligned} X = \frac{v_X}{\sqrt{2}\,\mu_X^3}\frac{1}{\sqrt{2}}\,\bar\psi &\left(\left(\begin{array}{cc|cc} 1 & & & \\ & 0 & & \\ \hline & & 1 & \\ & & & 0 \end{array}\right), \gamma^5\left(\begin{array}{cc|cc} 1 & & & \\ & 0 & & \\ \hline & & -1 & \\ & & & 0 \end{array}\right), 2\gamma^5\left(\begin{array}{cc|cc} & & 1 & \\ & & & 0 \\ \hline & & & \\ & & & \end{array}\right), 2\gamma^5\left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline 1 & & & \\ & 0 & & \end{array}\right)\right)\psi \\ = (X^0, X^3, X^+, X^-) \quad with \quad \mu_X^3 = \frac{<\bar u u + \bar d d>}{\sqrt{2}}, & \end{aligned}} \tag{4.1}$$

$$\boxed{\begin{aligned} H = \frac{v_H}{\sqrt{2}\,\mu_H^3}\frac{1}{\sqrt{2}}\,\bar\psi &\left(\left(\begin{array}{cc|cc} 0 & & & \\ & 1 & & \\ \hline & & 0 & \\ & & & 1 \end{array}\right), \gamma^5\left(\begin{array}{cc|cc} 0 & & & \\ & 1 & & \\ \hline & & 0 & \\ & & & -1 \end{array}\right), 2\gamma^5\left(\begin{array}{cc|cc} & & 0 & \\ & & & 1 \\ \hline & & & \\ & & & \end{array}\right), 2\gamma^5\left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline 0 & & & \\ & 1 & & \end{array}\right)\right)\psi \\ = (H^0, H^3, H^+, H^-) \quad with \quad \mu_H^3 = \frac{<\bar c c + \bar s s>}{\sqrt{2}}, & \end{aligned}} \tag{4.2}$$

$$
\boxed{\begin{aligned}\Omega = \frac{v_\Omega}{\sqrt{2}\,\mu_\Omega^3}\frac{1}{2}\,\bar\psi\left(\left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & 1 \\ & & 1 & \end{array}\right), \gamma^5\left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & -1 \\ & & -1 & \end{array}\right), 2\gamma^5\left(\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & & & \\ & & & \end{array}\right), 2\gamma^5\left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline & 1 & & \\ 1 & & & \end{array}\right)\right)\psi \\ = (\Omega^0, \Omega^3, \Omega^+, \Omega^-) \quad with \quad \mu_\Omega^3 = \frac{<\bar u c + \bar c u + \bar d s + \bar s d>}{2},\end{aligned}}
\tag{4.3}
$$

$$
\boxed{\begin{aligned}\Xi = \frac{v_\Xi}{\sqrt{2}\,\mu_\Xi^3}\frac{1}{2}\,\bar\psi\left(\left(\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & & 1 \\ & & -1 & \end{array}\right), \gamma^5\left(\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & & -1 \\ & & 1 & \end{array}\right), 2\gamma^5\left(\begin{array}{cc|cc} & & & 1 \\ & & -1 & \\ \hline & & & \\ & & & \end{array}\right), 2\gamma^5\left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline & 1 & & \\ -1 & & & \end{array}\right)\right)\psi \\ = (\Xi^0, \Xi^3, \Xi^+, \Xi^-) \quad with \quad \mu_\Xi^3 = \frac{<\bar u c - \bar c u + \bar d s - \bar s d>}{2},\end{aligned}}
\tag{4.4}
$$

to which are added their 4 parity-transformed alter egos of the type $(\mathfrak{p}^0, \vec{\mathfrak{s}})$, that we call $\hat X, \hat H, \hat\Omega, \hat\Xi$. Since the algebraic aspect of the last four is the same, but for the place of the γ_5 matrix, as that of X, H, Ω, Ξ, we omit writing them explicitly. To $\hat X, \hat H, \hat\Omega, \hat\Xi$ are associated the bosonic VEV's $\hat v_X, \hat v_H, \hat v_\Omega, \hat v_\Xi$, and the fermionic VEV's

$$
\boxed{\hat\mu_X^3 = \frac{<\bar u u - \bar d d>}{\sqrt{2}}, \hat\mu_H^3 = \frac{<\bar c c - \bar s s>}{\sqrt{2}}, \hat\mu_\Omega^3 = \frac{<\bar u c + \bar c u - \bar d s - \bar s d>}{2}, \hat\mu_\Xi^3 = \frac{<\bar u c - \bar c u - \bar d s + \bar s d>}{2}}
\tag{4.5}
$$

The 8 Higgs bosons are the 8 scalars $[X^0]$, $[H^0]$, $[\Omega^0]$, $[\Xi^0]$, $[\hat X^3]$, $[\hat H^3]$, $[\hat\Omega^3]$, $[\hat\Xi^3]$, and one has a total of $2\times 2N^2 = 16$ VEV's to determine: 8 v's and 8 μ^3's.

4.1.1 Choosing the quasi-standard Higgs doublet

We have now to choose which quadruplet includes the 3 Goldstone bosons of the spontaneously broken $SU(2)_L$, knowing that they will disappear from the mesonic spectrum.

It is not good a priori that the charged Goldstones are pseudoscalars, because charged pseudoscalar mesons are observed without ambiguity. This excludes all $(\mathfrak{s}, \vec{\mathfrak{p}})$-like ("unhatted") quadruplets. Among $\hat X, \hat H, \hat\Omega, \hat\Xi$ it is easy to see that the neutral pseudoscalar component of the last two are combinations of neutral K and D mesons, that we also want to keep in the mesonic spectrum. So, the only 2 left possibilities are $\hat X$ and $\hat H$. We choose $\hat H$ because it involves the heavy quark c and because it looks more natural to eliminate from the spectrum $\bar c\gamma_5 c + \bar s\gamma_5 s$ rather than $\bar u\gamma_5 u + \bar d\gamma_5 d$ which is related to observed low mass pseudoscalar mesons. The reader can argue that η_c is also observed, but we have in mind that, for 3 generations, there is not so much problem in eliminating from the spectrum some pseudoscalar singlet including $\bar t\gamma_5 t$.

We accordingly postulate that, for 2 generations

$$
\boxed{W_\parallel^+ \sim \hat H^+ \sim \bar c s\;(scalar), W_\parallel^- \sim \hat H^- \sim \bar s c\;(scalar), W_\parallel^3 \sim \hat H^0 \sim \bar c\gamma_5 c + \bar s\gamma_5 s\;(pseudoscalar)}
\tag{4.6}
$$

4.1.2 Notations

For a systematic and not too tedious treatment of the set of equations that will follow, we introduce, as already mentioned in (2.32), the following dimensionless ratios of bosonic VEV's, normalized to the one of the "quasi-standard" Higgs multiplet $\hat{H}$ ($\hat{b}_H$ is not present simply because it is identical to 1 by definition)

$$\boxed{\begin{array}{l} b_X \equiv \left(\dfrac{v_X}{\hat{v}_H}\right)^2, \quad b_H \equiv \left(\dfrac{v_H}{\hat{v}_H}\right)^2, \quad b_\Omega \equiv \left(\dfrac{v_\Omega}{\hat{v}_H}\right)^2, \quad b_\Xi \equiv \left(\dfrac{v_\Xi}{\hat{v}_H}\right)^2, \\ \hat{b}_X \equiv \left(\dfrac{\hat{v}_X}{\hat{v}_H}\right)^2, \quad \hat{b}_\Omega \equiv \left(\dfrac{\hat{v}_\Omega}{\hat{v}_H}\right)^2, \quad \hat{b}_\Xi \equiv \left(\dfrac{\hat{v}_\Xi}{\hat{v}_H}\right)^2. \end{array}} \tag{4.7}$$

Likewise we introduce dimensionless ratios of fermionic VEV's, normalized this time to $\mu_X^3 = \frac{<\bar{u}u+\bar{d}d>}{\sqrt{2}}$ ($r_X = 1$ by definition)

$$\boxed{\begin{array}{rrrr} & r_H \equiv \dfrac{\mu_H^3}{\mu_X^2}, & r_\Omega \equiv \dfrac{\mu_\Omega^3}{\mu_X^2}, & r_\Xi \equiv \dfrac{\mu_\Xi^3}{\mu_X^3}, \\ \hat{r}_X \equiv \dfrac{\hat{\mu}_X^3}{\mu_X^3}, & \hat{r}_H \equiv \dfrac{\hat{\mu}_H^3}{\mu_X^3}, & \hat{r}_\Omega \equiv \dfrac{\hat{\mu}_\Omega^3}{\mu_X^3}, & \hat{r}_\Xi \equiv \dfrac{\hat{\mu}_\Xi^3}{\mu_X^3}. \end{array}} \tag{4.8}$$

To lighten forthcoming equations, we shall also introduce the following ratios of bosonic to fermionic VEV's

$$\boxed{\frac{1}{\nu_i^2} \equiv \frac{v_i}{\sqrt{2}\mu_i^3}, \quad \frac{1}{\bar{\nu}_i^4} \equiv \frac{1-b_i}{\nu_i^4}, \quad \frac{1}{\hat{\nu}_i^2} \equiv \frac{\hat{v}_i}{\sqrt{2}\hat{\mu}_i^3}, \quad \frac{1}{\bar{\hat{\nu}}_i^{4}} \equiv \frac{1-\hat{b}_i}{\hat{\nu}_i^4}} \tag{4.9}$$

4.2 The kinetic terms and the mass of the gauge bosons

The kinetic terms are simply chosen to be a diagonal sum of the kinetic terms for each of the 8 quadruplets

$$\sum_{i\in[X,\hat{X},H,\hat{H},\Omega,\hat{\Omega},\Xi,\hat{\Xi}]} (D_\mu \Delta_i)^\dagger (D^\mu \Delta_i), \tag{4.10}$$

in which D_μ is the covariant derivative with respect to the $SU(2)_L$ group.

From (4.10) one gets

$$\boxed{m_W^2 = \frac{g^2}{4}\hat{v}_H^2(b_X + b_H + b_\Omega + b_\Xi + \hat{b}_X + \hat{b}_H + \hat{b}_\Omega + \hat{b}_\Xi)} \tag{4.11}$$

We recall that $\hat{b}_H = 1$ by definition. We shall also see in the next chapter (eqs. (5.4) and (5.22)) that one can take $b_\Xi = 0 = \hat{b}_\Xi$ as a consequence of $< \bar{u}c >=< \bar{c}u >$ and $< \bar{d}s >=< \bar{s}d >$.

4.3 Yukawa couplings and the Higgs potential

$\mathcal{L}_{Yuk}$ is of the general form (2.23) and must, by the choice (4.6), satisfy $\hat{\delta}_H = 0$ (see (2.35)). In order to avoid classical transitions between charged scalars and pseudoscalars, it must also satisfy the 4 equations $\delta_{i\hat{i}} + \kappa_{\hat{i}i} = 0, i \in \{X, H, \Omega, \Xi\}$ in (2.25). This leaves 11 independent Yukawa couplings to determine.

All VEV's, the b's, $\hat{b}$'s and the μ^3's, $\hat{\mu}^3$'s are a priori considered to be real. We shall see that this is well supported by the solutions of our equations, but for a problem concerning $\hat{\mu}_X^3 \equiv \frac{<\bar{u}u-\bar{d}d>}{\sqrt{2}}$.

The genuine scalar potential is given by (2.26) in which i spans the set $\{X, H, \Omega, \Xi\}$. It only depends on 2 parameters, m_H^2 and λ_H. Subtracting from it the bosonised form of the Yukawa Lagrangian yields the effective scalar potential V_{eff} which determines the vacuum of the theory.

Since $\hat{\delta}_H = 0$ the effective potential V_{eff} for the Higgs multiplet $\hat{H}$ is, by (2.25), identical to its genuine potential V. It has accordingly its minimum at $\hat{v}_H = v_0$ where v_0 has been defined in (2.28).

We also recall (see (2.31)) that the other equations defining the minima of V_{eff} entail $\delta_i = \delta(1-b_i), \quad \hat{\delta}_i = \delta(1-\hat{b}_i)$.

Note that both Yukawa couplings and the scalar potential can be written sums over pairs of parity-transformed quadruplets $(\Delta_i, \hat{\Delta}_i)$, the index i spanning accordingly the set $\{X, H, \Omega, \Xi\}$.

4.4 Charged pseudoscalar mesons

The Yukawa couplings induce, through the non-vanishing VEV's of $[X^0]$ and $[\hat{H}^3]$ non-diagonal fermionic mass terms $\bar{u}c, \bar{c}u, \bar{d}s, \bar{s}d$. As usual, the diagonalization of the two mass matrices leads to the quark mass eigenstates u_m, c_m, d_m, s_m which are connected to flavor eigenstates by two rotations with respective angles θ_u and θ_d (c_u, s_u mean respectively $\cos\theta_u, \sin\theta_u$ etc)

$$\begin{pmatrix} u \\ c \end{pmatrix} = \begin{pmatrix} c_u & s_u \\ -s_u & c_u \end{pmatrix} \begin{pmatrix} u_m \\ c_m \end{pmatrix}, \quad \begin{pmatrix} d \\ s \end{pmatrix} = \begin{pmatrix} c_d & s_d \\ -s_d & c_d \end{pmatrix} \begin{pmatrix} d_m \\ s_m \end{pmatrix}, \tag{4.12}$$

At this stage of our study, we shall not investigate the link between Yukawa couplings and mixing angles, leaving this for section 5.6. All that we need for the present purpose is to re-express the flavor quark bilinears in terms of bilinears of quark mass eigenstates by using (4.12). The corresponding formulæ, which are used throughout the paper, have been gathered in Appendix .1.

From now onwards, the connexion between mesonic fields and bilinear quark operators, that is low energy relations (PCAC, GMOR), is done through $\bar{q}_m^i \gamma_5 q_m^j$ or $\bar{q}_m^i q_m^j$ involving quark mass eigenstates.

The main result of this section, the mixing formula (1.8), will be obtained from the following very simple statements concerning the charged $\pi^\pm, K^\pm, D^\pm, D_s^\pm$ pseudoscalar mesons only:

* their $mass^2$ are obtained from the appropriate ratios between the Yukawa couplings (as explained in subsection 2.3.3) and kinetic terms; for example, for charged pions, one selects in $\mathcal{L}_{Yuk}$ and in $\mathcal{L}_{kin}$ all terms proportional to $(\bar{u}_m\gamma_5 d_m)(\bar{d}_m\gamma_5 u_m)$, eventually using the twofold nature of the Higgs multiplets to express any bosonic components in terms of quark fields;

* one demands that no non-diagonal transition occurs between any of them; for example one cancels in $\mathcal{L}_{Yuk}$ the coefficient of the $(\bar{u}_m\gamma_5 d_m)(\bar{s}_m\gamma_5 u_m)$ terms which would correspond to transitions between π^+ and K^+.

All normalization factors that arise from low energy theorems (PCAC, GMOR) cancel (in the ratios defining masses) or are irrelevant (when setting some coefficient to zero). This entails in particular that eq. (1.8) does not depend on low energy theorems but only on the much weaker hypothesis that charged mesonic π^+, K^+, D^+, D_s^+ fields are respectively proportional to $\bar{u}_m\gamma_5 d_m, \bar{u}_m\gamma_5 s_m, \bar{c}_m\gamma_5 d_m, \bar{c}_m\gamma_5 s_m$. It is therefore a robust result.

4.4.1 Orthogonality

Charged pseudoscalar mesons can only be found in the "non-hatted" quadruplets X, H, Ω, Ξ and their quadratics can only be found in the terms $-\delta_X X^\dagger X - \delta_H H^\dagger H - \delta_\Omega \Omega^\dagger \Omega - \delta_\Xi \Xi^\dagger \Xi$ of $\mathcal{L}_{Yuk}$.

Using Appendix .1, the 6 equations expressing the vanishing of non-diagonal transitions among them are found to be the following

$$
\begin{aligned}
\pi^{\pm} \not\leftrightarrow K^{\pm} &\Leftrightarrow \delta_X \frac{c_u c_d c_u s_d}{\nu_X^4} - \delta_H \frac{s_u s_d s_u c_d}{\nu_H^4} - \frac{\delta_\Omega}{2} \frac{s_{u+d} c_{u+d}}{\nu_\Omega^4} + \frac{\delta_\Xi}{2} \frac{s_{u-d} c_{u-d}}{\nu_\Xi^4} = 0,\\
\pi^{\pm} \not\leftrightarrow D^{\pm} &\Leftrightarrow \delta_X \frac{c_u c_d s_u c_d}{\nu_X^4} - \delta_H \frac{s_u s_d c_u s_d}{\nu_H^4} - \frac{\delta_\Omega}{2} \frac{s_{u+d} c_{u+d}}{\nu_\Omega^4} - \frac{\delta_\Xi}{2} \frac{s_{u-d} c_{u-d}}{\nu_\Xi^4} = 0,\\
\pi^{\pm} \not\leftrightarrow D_s^{\pm} &\Leftrightarrow \delta_X \frac{s_u s_d c_u c_d}{\nu_X^4} + \delta_H \frac{s_u s_d c_u c_d}{\nu_H^4} - \frac{\delta_\Omega}{2} \frac{s_{u+d}^2}{\nu_\Omega^4} + \frac{\delta_\Xi}{2} \frac{s_{u-d}^2}{\nu_\Xi^4} = 0,\\
K^{\pm} \not\leftrightarrow D^{\pm} &\Leftrightarrow \delta_X \frac{c_u s_d s_u c_d}{\nu_X^4} + \delta_H \frac{s_u c_d c_u s_d}{\nu_H^4} + \frac{\delta_\Omega}{2} \frac{c_{u+d}^2}{\nu_\Omega^4} - \frac{\delta_\Xi}{2} \frac{c_{u-d}^2}{\nu_\Xi^4} = 0,\\
K^{\pm} \not\leftrightarrow D_s^{\pm} &\Leftrightarrow \delta_X \frac{c_u s_d s_u s_d}{\nu_X^4} - \delta_H \frac{s_u c_d c_u c_d}{\nu_H^4} + \frac{\delta_\Omega}{2} \frac{s_{u+d} c_{u+d}}{\nu_\Omega^4} + \frac{\delta_\Xi}{2} \frac{s_{u-d} c_{u-d}}{\nu_\Xi^4} = 0,\\
D^{\pm} \not\leftrightarrow D_s^{\pm} &\Leftrightarrow \delta_X \frac{s_u c_d s_u s_d}{\nu_X^4} - \delta_H \frac{c_u s_d c_u c_d}{\nu_H^4} + \frac{\delta_\Omega}{2} \frac{s_{u+d} c_{u+d}}{\nu_\Omega^4} - \frac{\delta_\Xi}{2} \frac{s_{u-d} c_{u-d}}{\nu_\Xi^4} = 0.
\end{aligned}
\tag{4.13}
$$

in which c_{u-d} stands for $\cos(\theta_u - \theta_d)$ *etc* .

Using the relations (2.31) allows to factor out δ. Then using the notations (4.9) transforms the 6 equations of (4.13) respectively into

$$
\begin{aligned}
(a) &: \frac{c_u c_d c_u s_d}{\bar\nu_X^4} - \frac{s_u s_d s_u c_d}{\bar\nu_H^4} - \frac{1}{2} \frac{s_{u+d}\, c_{u+d}}{\bar\nu_\Omega^4} + \frac{1}{2} \frac{s_{u-d}\, c_{u-d}}{\bar\nu_\Xi^4} = 0,\\
(b) &: \frac{c_u c_d s_u c_d}{\bar\nu_X^4} - \frac{s_u s_d c_u s_d}{\bar\nu_H^4} - \frac{1}{2} \frac{s_{u+d}\, c_{u+d}}{\bar\nu_\Omega^4} - \frac{1}{2} \frac{s_{u-d}\, c_{u-d}}{\bar\nu_\Xi^4} = 0,\\
(c) &: \frac{s_u s_d c_u c_d}{\bar\nu_X^4} + \frac{s_u s_d c_u c_d}{\bar\nu_H^4} - \frac{1}{2} \frac{s_{u+d}^2}{\bar\nu_\Omega^4} + \frac{1}{2} \frac{s_{u-d}^2}{\bar\nu_\Xi^4} = 0,\\
(d) &: \frac{c_u s_d s_u c_d}{\bar\nu_X^4} + \frac{s_u c_d c_u s_d}{\bar\nu_H^4} + \frac{1}{2} \frac{c_{u+d}^2}{\bar\nu_\Omega^4} - \frac{1}{2} \frac{c_{u-d}^2}{\bar\nu_\Xi^4} = 0,\\
(e) &: \frac{c_u s_d s_u s_d}{\bar\nu_X^4} - \frac{s_u c_d c_u c_d}{\bar\nu_H^4} + \frac{1}{2} \frac{s_{u+d}\, c_{u+d}}{\bar\nu_\Omega^4} + \frac{1}{2} \frac{s_{u-d}\, c_{u-d}}{\bar\nu_\Xi^4} = 0,\\
(f) &: \frac{s_u c_d s_u s_d}{\bar\nu_X^4} - \frac{c_u s_d c_u c_d}{\bar\nu_H^4} + \frac{1}{2} \frac{s_{u+d}\, c_{u+d}}{\bar\nu_\Omega^4} - \frac{1}{2} \frac{s_{u-d}\, c_{u-d}}{\bar\nu_\Xi^4} = 0,
\end{aligned}
\tag{4.14}
$$

or, equivalently, by recombining them,

$$
\begin{aligned}
(a)+(f) &: s_{2d} \left(\frac{1}{\bar\nu_X^4} - \frac{1}{\bar\nu_H^4} \right) = 0,\\
(a)-(f) &: s_{2d} c_{2u} \left(\frac{1}{\bar\nu_X^4} + \frac{1}{\bar\nu_H^4} \right) - \frac{s_{2(u+d)}}{\bar\nu_\Omega^4} + \frac{s_{2(u-d)}}{\bar\nu_\Xi^4} = 0,\\
(b)-(e) &: s_{2u} c_{2d} \left(\frac{1}{\bar\nu_X^4} + \frac{1}{\bar\nu_H^4} \right) - \frac{s_{2(u+d)}}{\bar\nu_\Omega^4} - \frac{s_{2(u-d)}}{\bar\nu_\Xi^4} = 0,\\
(b)+(e) &: s_{2u} \left(\frac{1}{\bar\nu_X^4} - \frac{1}{\bar\nu_H^4} \right) = 0,\\
(c)-(d) &: \frac{1}{\bar\nu_\Omega^4} - \frac{1}{\bar\nu_\Xi^4} = 0,\\
(c)+(d) &: s_{2u} s_{2d} \left(\frac{1}{\bar\nu_X^4} + \frac{1}{\bar\nu_H^4} \right) + \frac{c_{2(u+d)}}{\bar\nu_\Omega^4} - \frac{c_{2(u-d)}}{\bar\nu_\Xi^4} = 0.
\end{aligned}
\tag{4.15}
$$

The solution of (4.15) is

$$
\frac{1}{\bar\nu_X^4} = \frac{1}{\bar\nu_H^4} = \frac{1}{\bar\nu_\Omega^4} = \frac{1}{\bar\nu_\Xi^4} \qquad \overset{(4.9)}{\Leftrightarrow} \qquad \frac{1-b_X}{\nu_X^4} = \frac{1-b_H}{\nu_H^4} = \frac{1-b_\Omega}{\nu_\Omega^4} = \frac{1-b_\Xi}{\nu_\Xi^4}. \tag{4.16}
$$

Using the definitions of the b's and ν^2's given in (2.32) and (4.9), (4.16) also writes

$$
\boxed{\frac{b_X(1-b_X)}{\mu_X^6} = \frac{b_H(1-b_H)}{\mu_H^6} = \frac{b_\Omega(1-b_\Omega)}{\mu_\Omega^6} = \frac{b_\Xi(1-b_\Xi)}{\mu_\Xi^6}} \tag{4.17}
$$

Note that we only took into account the non-diagonal quadratic terms that occur in the Yukawa Lagrangian. Such terms are also present in the kinetic Lagrangian and the conditions for their vanishing are not the same as (4.13). They however vanish at small momentum such that the solution (4.16) (4.17) can only be considered to be valid at this limit.

4.4.2 Masses

From the ratios of the terms quadratic in the meson fields in the Yukawa and kinetic terms, using (2.31) and the notation (2.32) one gets, with the help of Appendix .1

$$
\begin{aligned}
m^2_{\pi^\pm} &= \delta \frac{(1-b_X)\,(\frac{c_u c_d}{\nu_X^2})^2 + (1-b_H)\,(\frac{s_u s_d}{\nu_H^2})^2 + (1-b_\Omega)\,\frac{1}{2}(\frac{s_{u+d}}{\nu_\Omega^2})^2 + (1-b_\Xi)\,\frac{1}{2}(\frac{s_{u-d}}{\nu_\Xi^2})^2}{(\frac{c_u c_d}{\nu_X^2})^2 + (\frac{s_u s_d}{\nu_H^2})^2 + \frac{1}{2}(\frac{s_{u+d}}{\nu_\Omega^2})^2 + \frac{1}{2}(\frac{s_{u-d}}{\nu_\Xi^2})^2},\\
m^2_{K^\pm} &= \delta \frac{(1-b_X)\,(\frac{c_u s_d}{\nu_X^2})^2 + (1-b_H)\,(\frac{s_u c_d}{\nu_H^2})^2 + (1-b_\Omega)\,\frac{1}{2}(\frac{c_{u+d}}{\nu_\Omega^2})^2 + (1-b_\Xi)\,\frac{1}{2}(\frac{c_{u-d}}{\nu_\Xi^2})^2}{(\frac{c_u s_d}{\nu_X^2})^2 + (\frac{s_u c_d}{\nu_H^2})^2 + \frac{1}{2}(\frac{c_{u+d}}{\nu_\Omega^2})^2 + \frac{1}{2}(\frac{c_{u-d}}{\nu_\Xi^2})^2},\\
m^2_{D^\pm} &= \delta \frac{(1-b_X)\,(\frac{s_u c_d}{\nu_X^2})^2 + (1-b_H)\,(\frac{c_u s_d}{\nu_H^2})^2 + (1-b_\Omega)\,\frac{1}{2}(\frac{c_{u+d}}{\nu_\Omega^2})^2 + (1-b_\Xi)\,\frac{1}{2}(\frac{c_{u-d}}{\nu_\Xi^2})^2}{(\frac{s_u c_d}{\nu_X^2})^2 + (\frac{c_u s_d}{\nu_H^2})^2 + \frac{1}{2}(\frac{c_{u+d}}{\nu_\Omega^2})^2 + \frac{1}{2}(\frac{c_{u-d}}{\nu_\Xi^2})^2},\\
m^2_{D_s^\pm} &= \delta \frac{(1-b_X)\,(\frac{s_u s_d}{\nu_X^2})^2 + (1-b_H)\,(\frac{c_u c_d}{\nu_H^2})^2 + (1-b_\Omega)\,\frac{1}{2}(\frac{s_{u+d}}{\nu_\Omega^2})^2 + (1-b_\Xi)\,\frac{1}{2}(\frac{s_{u-d}}{\nu_\Xi^2})^2}{(\frac{s_u s_d}{\nu_X^2})^2 + (\frac{c_u c_d}{\nu_H^2})^2 + \frac{1}{2}(\frac{s_{u+d}}{\nu_\Omega^2})^2 + \frac{1}{2}(\frac{s_{u-d}}{\nu_\Xi^2})^2},
\end{aligned}
\tag{4.18}
$$

which rewrites, using (4.16)

$$
\begin{aligned}
m^2_{\pi^\pm} &= \frac{\delta/\bar\nu_X^4}{(c_u c_d/\nu_X^2)^2 + (s_u s_d/\nu_H^2)^2 + \frac{1}{2}(s_{u+d}/\nu_\Omega^2)^2 + \frac{1}{2}(s_{u-d}/\nu_\Xi^2)^2},\\
m^2_{K^\pm} &= \frac{\delta/\bar\nu_X^4}{(c_u s_d/\nu_X^2)^2 + (s_u c_d/\nu_H^2)^2 + \frac{1}{2}(c_{u+d}/\nu_\Omega^2)^2 + \frac{1}{2}(c_{u-d}/\nu_\Xi^2)^2},\\
m^2_{D^\pm} &= \frac{\delta/\bar\nu_X^4}{(s_u c_d/\nu_X^2)^2 + (c_u s_d/\nu_H^2)^2 + \frac{1}{2}(c_{u+d}/\nu_\Omega^2)^2 + \frac{1}{2}(c_{u-d}/\nu_\Xi^2)^2},\\
m^2_{D_s^\pm} &= \frac{\delta/\bar\nu_X^4}{(s_u s_d/\nu_X^2)^2 + (c_u c_d/\nu_H^2)^2 + \frac{1}{2}(s_{u+d}/\nu_\Omega^2)^2 + \frac{1}{2}(s_{u-d}/\nu_\Xi^2)^2}.
\end{aligned}
\tag{4.19}
$$

Recombining the 4 equations in (4.19) yields

$$
\begin{aligned}
&\delta\left(+\frac{1}{m^2_{\pi^\pm}} + \frac{1}{m^2_{K^\pm}} + \frac{1}{m^2_{D^\pm}} + \frac{1}{m^2_{D_s^\pm}}\right) = \frac{1}{1-b_X} + \frac{1}{1-b_H} + \frac{1}{1-b_\Omega} + \frac{1}{1-b_\Xi},\\
&\delta\left(+\frac{1}{m^2_{\pi^\pm}} - \frac{1}{m^2_{K^\pm}} + \frac{1}{m^2_{D^\pm}} - \frac{1}{m^2_{D_s^\pm}}\right) = c_{2d}\left(\frac{1}{1-b_X} - \frac{1}{1-b_H}\right),\\
&\delta\left(+\frac{1}{m^2_{\pi^\pm}} + \frac{1}{m^2_{K^\pm}} - \frac{1}{m^2_{D^\pm}} - \frac{1}{m^2_{D_s^\pm}}\right) = c_{2u}\left(\frac{1}{1-b_X} - \frac{1}{1-b_H}\right),\\
&\delta\left(+\frac{1}{m^2_{\pi^\pm}} - \frac{1}{m^2_{K^\pm}} - \frac{1}{m^2_{D^\pm}} + \frac{1}{m^2_{D_s^\pm}}\right) = c_{2u}c_{2d}\left(\frac{1}{1-b_X} + \frac{1}{1-b_H}\right) - \frac{c_{2(u+d)}}{1-b_\Omega} - \frac{c_{2(u-d)}}{1-b_\Xi}.
\end{aligned}
\tag{4.20}
$$

4.5 Mixing : the master formula

From the second and third equations of (4.20) one gets, independently of the scale δ

$$
\boxed{\frac{c_{2u}-c_{2d}}{c_{2u}+c_{2d}} \equiv \tan(\theta_d+\theta_u)\tan(\theta_d-\theta_u) = \frac{\dfrac{1}{m^2_{K^\pm}} - \dfrac{1}{m^2_{D^\pm}}}{\dfrac{1}{m^2_{\pi^\pm}} - \dfrac{1}{m^2_{D_s^\pm}}}}
\tag{4.21}
$$

which vanishes either at the chiral limit $m_\pi \to 0$ or when $m_K = m_D$. (4.21) is independent of all the VEV's. As we have already mentioned, all normalization factors that occur in PCAC or GMOR relations cancel out [1]. A major property of (4.21) is that θ_u and θ_d are not independent variables. In particular, the positivity of its r.h.s. entails that the spectrum of charged pseudoscalar mesons is only compatible with $\theta_u < \theta_d$.

4.6 Neutral pseudoscalar mesons

We shall define hereafter the interpolating fields of π^0 and η to be respectively proportional to $\bar{u}\gamma_5 u - \bar{d}\gamma_5 d$ and $\bar{u}\gamma_5 u + \bar{d}\gamma_5 d$, which does not exactly corresponds to the divergences of the axial currents $\bar{u}\gamma^\mu\gamma_5 u - \bar{d}\gamma^\mu\gamma_5 d$ and $\bar{u}\gamma^\mu\gamma_5 u + \bar{d}\gamma^\mu\gamma_5 d$. We make this choice because of the negative sign that comes out for the d quark mass; we shall see that it leads to suitable orthogonality relations and mass for the π^0, and that only η looks somewhat more problematic. Deeper investigations concerning this states are postponed to further works, and we shall not use η in the present one to fit the parameters.

[1]Eq. (4.21) appears like a generalization of the result by Oakes [8] which used the hypothesis that chiral $SU(2) \times SU(2)$ breaking (by strong interactions) and the non-conservation of strangeness (Cabibbo angle) in weak interactions have a common origin.

4.6.1 Orthogonality

The equations are the following

$$\boxed{\begin{array}{l}
(a)\ D^0 \perp K^0 : s_u c_u s_d c_d(-\frac{\delta_X}{\nu_X^4}+\frac{\hat\delta_X}{\hat\nu_X^4}-\frac{\delta_H}{\nu_H^4}+\frac{\hat\delta_H}{\hat\nu_H^4})+\frac12 c_{2u}c_{2d}(-\frac{\delta_\Omega}{\nu_\Omega^4}+\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4})+\frac12(-\frac{\delta_\Xi}{\nu_\Xi^4}+\frac{\hat\delta_\Xi}{\hat\nu_\Xi^4})=0,\\
(b)\ \bar D^0 \perp \bar K^0 : idem,\\
(c)\ D^0 \perp \bar K^0 : s_u c_u s_d c_d(-\frac{\delta_X}{\nu_X^4}+\frac{\hat\delta_X}{\hat\nu_X^4}-\frac{\delta_H}{\nu_H^4}+\frac{\hat\delta_H}{\hat\nu_H^4})+\frac12 c_{2u}c_{2d}(-\frac{\delta_\Omega}{\nu_\Omega^4}+\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4})-\frac12(-\frac{\delta_\Xi}{\nu_\Xi^4}+\frac{\hat\delta_\Xi}{\hat\nu_\Xi^4})=0,\\
(d)\ \bar D^0 \perp K^0 : idem,\\
(e)\ \pi^0 \perp (D^0+\bar D^0) : s_u c_u\frac{c_u^2+c_d^2}{2}\frac{\delta_X}{\nu_X^4}+s_u c_u\frac{c_u^2-c_d^2}{2}\frac{\hat\delta_X}{\hat\nu_X^4}-s_u c_u\frac{s_u^2+s_d^2}{2}\frac{\delta_H}{\nu_H^4}-s_u c_u\frac{s_u^2-s_d^2}{2}\frac{\hat\delta_H}{\hat\nu_H^4}\\
\qquad\qquad -\frac12 c_{2u}\frac{s_{2u}+s_{2d}}{2}\frac{\delta_\Omega}{\nu_\Omega^4}-\frac12 c_{2u}\frac{s_{2u}-s_{2d}}{2}\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4}=0,\\
(f)\ \eta \perp (D^0+\bar D^0) : s_u c_u\frac{c_u^2-c_d^2}{2}\frac{\delta_X}{\nu_X^4}+s_u c_u\frac{c_u^2+c_d^2}{2}\frac{\hat\delta_X}{\hat\nu_X^4}-s_u c_u\frac{s_u^2-s_d^2}{2}\frac{\delta_H}{\nu_H^4}-s_u c_u\frac{s_u^2+s_d^2}{2}\frac{\hat\delta_H}{\hat\nu_H^4}\\
\qquad\qquad -\frac12 c_{2u}\frac{s_{2u}-s_{2d}}{2}\frac{\delta_\Omega}{\nu_\Omega^4}-\frac12 c_{2u}\frac{s_{2u}+s_{2d}}{2}\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4}=0,\\
\pi^0 \perp (K^0-\bar K^0) : always\ true,\\
\eta \perp (K^0-\bar K^0) : always\ true,\\
\pi^0 \perp (D^0-\bar D^0) : always\ true,\\
\eta \perp (D^0-\bar D^0) : always\ true,\\
(K^0+\bar K^0) \perp (K^0-\bar K^0) : always\ true,\\
(D^0+\bar D^0) \perp (D^0-\bar D^0) : always\ true,\\
(g)\ \pi^0 \perp (K^0+\bar K^0) : -s_d c_d\frac{c_u^2+c_d^2}{2}\frac{\delta_X}{\nu_X^4}+s_d c_d\frac{c_u^2-c_d^2}{2}\frac{\hat\delta_X}{\hat\nu_X^4}+s_d c_d\frac{s_u^2+s_d^2}{2}\frac{\delta_H}{\nu_H^4}-s_d c_d\frac{s_u^2-s_d^2}{2}\frac{\hat\delta_H}{\hat\nu_H^4}\\
\qquad\qquad +\frac12 c_{2d}\frac{s_{2u}+s_{2d}}{2}\frac{\delta_\Omega}{\nu_\Omega^4}-\frac12 c_{2d}\frac{s_{2u}-s_{2d}}{2}\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4}=0,\\
(h)\ \eta \perp (K^0+\bar K^0) : -s_d c_d\frac{c_u^2-c_d^2}{2}\frac{\delta_X}{\nu_X^4}+s_d c_d\frac{c_u^2+c_d^2}{2}\frac{\hat\delta_X}{\hat\nu_X^4}+s_d c_d\frac{s_u^2-s_d^2}{2}\frac{\delta_H}{\nu_H^4}-s_d c_d\frac{s_u^2+s_d^2}{2}\frac{\hat\delta_H}{\hat\nu_H^4}\\
\qquad\qquad +\frac12 c_{2d}\frac{s_{2u}-s_{2d}}{2}\frac{\delta_\Omega}{\nu_\Omega^4}-\frac12 c_{2d}\frac{s_{2u}+s_{2d}}{2}\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4}=0,\\
(i)\ D^0 \perp \bar D^0 : s_u^2 c_u^2(\frac{\delta_X}{\nu_X^4}+\frac{\hat\delta_X}{\hat\nu_X^4}+\frac{\delta_H}{\nu_H^4}+\frac{\hat\delta_H}{\hat\nu_H^4})+\frac12 c_{2u}^2(\frac{\delta_\Omega}{\nu_\Omega^4}+\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4})-\frac12(\frac{\delta_\Xi}{\nu_\Xi^4}+\frac{\hat\delta_\Xi}{\hat\nu_\Xi^4})=0,\\
(j)\ K^0 \perp \bar K^0 : s_d^2 c_d^2(\frac{\delta_X}{\nu_X^4}+\frac{\hat\delta_X}{\hat\nu_X^4}+\frac{\delta_H}{\nu_H^4}+\frac{\hat\delta_H}{\hat\nu_H^4})+\frac12 c_{2d}^2(\frac{\delta_\Omega}{\nu_\Omega^4}+\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4})-\frac12(\frac{\delta_\Xi}{\nu_\Xi^4}+\frac{\hat\delta_\Xi}{\hat\nu_\Xi^4})=0.
\end{array}}
\tag{4.22}$$

4.6.2 Masses of π^0, K^0 and D^0

One gets

$$\boxed{m^2_{\pi^0}=\frac{\left(\frac{c_u^2+c_d^2}{2}\right)^2\frac{\delta_X}{\nu_X^4}+\left(\frac{c_u^2-c_d^2}{2}\right)^2\frac{\hat\delta_X}{\hat\nu_X^4}+\left(\frac{s_u^2+s_d^2}{2}\right)^2\frac{\delta_H}{\nu_H^4}+\left(\frac{s_u^2-s_d^2}{2}\right)^2\frac{\hat\delta_H}{\hat\nu_H^4}+\frac12\left(\frac{s_{2u}+s_{2d}}{2}\right)^2\frac{\delta_\Omega}{\nu_\Omega^4}+\frac12\left(\frac{s_{2u}-s_{2d}}{2}\right)^2\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4}}{\left(\frac{c_u^2+c_d^2}{2}\right)^2\frac{1}{\nu_X^4}+\left(\frac{c_u^2-c_d^2}{2}\right)^2\frac{1}{\hat\nu_X^4}+\left(\frac{s_u^2+s_d^2}{2}\right)^2\frac{1}{\nu_H^4}+\left(\frac{s_u^2-s_d^2}{2}\right)^2\frac{1}{\hat\nu_H^4}+\frac12\left(\frac{s_{2u}+s_{2d}}{2}\right)^2\frac{1}{\nu_\Omega^4}+\frac12\left(\frac{s_{2u}-s_{2d}}{2}\right)^2\frac{1}{\hat\nu_\Omega^4}}}
\tag{4.23}$$

$$\boxed{m^2_{K^0} = \frac{\frac{s_d^2c_d^2}{2}\left(\frac{\delta_X}{\nu_X^4}+\frac{\hat\delta_X}{\hat\nu_X^4}+\frac{\delta_H}{\nu_H^4}+\frac{\hat\delta_H}{\hat\nu_H^4}\right)+\frac{c_{2d}^2}{4}\left(\frac{\delta_\Omega}{\nu_\Omega^4}+\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4}\right)+\frac{1}{4}\left(\frac{\delta_\Xi}{\nu_\Xi^4}+\frac{\hat\delta_\Xi}{\hat\nu_\Xi^4}\right)}{\frac{s_d^2c_d^2}{2}\left(\frac{1}{\nu_X^4}+\frac{1}{\hat\nu_X^4}+\frac{1}{\nu_H^4}+\frac{1}{\hat\nu_H^4}\right)+\frac{c_{2d}^2}{4}\left(\frac{1}{\nu_\Omega^4}+\frac{1}{\hat\nu_\Omega^4}\right)+\frac{1}{4}\left(\frac{1}{\nu_\Xi^4}+\frac{1}{\hat\nu_\Xi^4}\right)}} \tag{4.24}$$

$$\boxed{m^2_{D^0} = \frac{\frac{s_u^2c_u^2}{2}\left(\frac{\delta_X}{\nu_X^4}+\frac{\hat\delta_X}{\hat\nu_X^4}+\frac{\delta_H}{\nu_H^4}+\frac{\hat\delta_H}{\hat\nu_H^4}\right)+\frac{c_{2u}^2}{4}\left(\frac{\delta_\Omega}{\nu_\Omega^4}+\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4}\right)+\frac{1}{4}\left(\frac{\delta_\Xi}{\nu_\Xi^4}+\frac{\hat\delta_\Xi}{\hat\nu_\Xi^4}\right)}{\frac{s_u^2c_u^2}{2}\left(\frac{1}{\nu_X^4}+\frac{1}{\hat\nu_X^4}+\frac{1}{\nu_H^4}+\frac{1}{\hat\nu_H^4}\right)+\frac{c_{2u}^2}{4}\left(\frac{1}{\nu_\Omega^4}+\frac{1}{\hat\nu_\Omega^4}\right)+\frac{1}{4}\left(\frac{1}{\nu_\Xi^4}+\frac{1}{\hat\nu_\Xi^4}\right)}} \tag{4.25}$$

We now have at our disposal all the tools to successively study the case when one approximates θ_u with 0, and the more general one when this approximation is relaxed.

4.7 Moving inside the space of Higgs multiplets

All quadruplets are sets of 4 elements which are stable both by $SU(2)_L$ and $SU(2)_R$, according to the laws of transformations (2.18) and (2.20).

There also exists a $U(2)^g_L \times U(2)^g_R$ group of transformations orthogonal to the former, that moves inside the 8-dimensional space of quadruplets. It includes the group $U(1)_L \times U(1)_R$ the generators of which swap parity and transform, for example, up to their normalizations and a sign, X into $\hat X$.

In the case of 2 generations the corresponding 2 sets of 4 generators are made of the 4×4 identity matrix and of the 3 following ones

$$L^1 = \frac{1}{2}\left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & 1 \\ & & 1 & \end{array}\right), \quad L^2 = -\frac{i}{2}\left(\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & & 1 \\ & & -1 & \end{array}\right), \quad L^3 = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & & & \\ & -1 & & \\ \hline & & 1 & \\ & & & -1 \end{array}\right), \tag{4.26}$$

which act on quark bilinears according to (2.12) and (2.14). They satisfy the following commutation and anticommutation relations

$$[L^i, L^j] = i\epsilon_{ijk}L^k, \quad \{L_i, L^j\} = 0, i \neq j. \tag{4.27}$$

The generators $\vec L$ given in (4.26) commute with the generators $\vec T$ of the gauge group given in (2.11)

$$[\vec L, \vec T] = 0, \tag{4.28}$$

which makes them “orthogonal” groups, while, for anticommutation

$$\{\vec L, \vec T\} = \vec L. \tag{4.29}$$

This second chiral group of transformations cannot be a symmetry of the theory as soon as the normalization factors $\frac{v}{\mu^3}$ are not identical. It gets thus broken by $v_i \neq v_j \neq \hat v_i \neq \hat v_j$ and $\mu_i^3 \neq \mu_j^3 \neq \hat\mu_i^3 \neq \hat\mu_j^3$.

In the case of 1 generation, L^1 and L^2 collapse to 0, such that only L^3 is left, which becomes proportional to the unit matrix. The chiral group under consideration therefore shrinks down to $U(1)_L \times U(1)_R$, in close connection, as we have seen in (2.22), with parity.

More information concerning this group and its close relation to the flipping of generations will be given in section 7.1.

Chapter 5

2 generations with $\theta_d \neq 0, \theta_u = 0$

It is usual in the GSW model to perform a flavor rotation on the (u, c) quarks to align their flavor and mass eigenstates. This is tantamount to setting $\theta_u = 0$ such that the Cabibbo angle $\theta_c = \theta_d - \theta_u$ becomes identical to θ_d. In there, it is allowed, as well, to keep instead $\theta_u \neq 0$ and to make a flavor rotation on (d, s) quarks to tune θ_d to 0. In the extension that we propose, the situation is different and setting $\theta_u = 0$ leads to problems that will only be cured at $\theta_u \neq 0$ (chapter 6).

5.1 Charged pseudoscalar mesons and the Cabibbo angle

Flavor rotations that would eventually tune a mixing angle to 0 should be operated on the fermion fields wherever they appear, which also includes all bilinear quark operators. In this respect, they can no longer be considered as "innocuous", all the more as there is no reason why they should correspond to an unbroken subgroup of $U(4)_L \times U(4)_R$.

We can thus only test the hypothesis that θ_u is small enough to be neglected with respect to θ_d, in which case $\tan(\theta_d + \theta_u)\tan(\theta_d - \theta_u) \simeq \tan^2\theta_d - \theta_u^2 \approx \tan^2\theta_d \approx \tan^2\theta_c$ and, from (4.21) "the" mixing angle is given approximately by

$$t^2 \equiv \tan^2\theta_d \approx \frac{\dfrac{1}{m^2_{K^\pm}} - \dfrac{1}{m^2_{D^\pm}}}{\dfrac{1}{m^2_{\pi^\pm}} - \dfrac{1}{m^2_{D_s^\pm}}} \approx \frac{m^2_{\pi^\pm}}{m^2_{K^\pm}}\left(1 - \frac{m^2_{K^\pm}}{m^2_{D^\pm}} + \frac{m^2_{\pi^\pm}}{m^2_{D_s^\pm}} + \mathcal{O}\Big(\frac{m^2_\pi, m^2_K}{m^2_D, m^2_{D_s}}\Big)^2\right). \tag{5.1}$$

Numerically, for the physical values of the charged pseudoscalar mesons [20]

$$m_{\pi^+} = 139.570\, MeV, \quad m_{K^+} = 493.677\, MeV, \quad m_{D^+} = 1.86962\, GeV, \quad m_{D_s^+} = 1.96849\, GeV. \tag{5.2}$$

one gets $t^2 \approx .07473$ which corresponds to

$$\theta_d \approx .26685, \tag{5.3}$$

to be compared with the experimental value (1.6). The value of the Cabibbo angle is not expected to be very sensitive to the existence of heavier generations; nevertheless, our result displays a 15% discrepancy. It can have two origins (if we forget about the absence of a 3rd generation). The first is that our result is only valid at the low momentum limit, at which non-diagonal transitions coming from kinetic terms vanish. While it is most probably an accurate limit for pions, its reliability is more questionable for heavier mesons. The second points at a non-vanishing value of θ_u, and indeed, in chapter 6, we will show that, indeed, θ_u plays a very important role.

5.2 Determination of $b_X, b_H, b_\Omega, b_\Xi$ in terms of δ

The b parameters have been defined in (4.7). First, we can take

$$\boxed{b_\Xi = 0} \tag{5.4}$$

Indeed, $\mu_\Xi^3 \equiv \dfrac{<\bar u c - \bar c u + \bar d s - \bar s d>}{2}$ can be considered to be vanishing at the limit when the vacuum is supposed to be invariant by C. The choice (5.4) corresponds to a finite normalization factor $\frac{v_\Xi}{\sqrt{2}\mu_\Xi^3}$ for the Ξ quadruplet. Using the definition (2.32) of b_Ξ, one can indeed write at this limit $\frac{v_\Xi}{\sqrt{2}\mu_\Xi^3} \equiv \sqrt{\frac{b_\Xi \hat v_H^2}{2\mu_\Xi^6}} \stackrel{b_\Xi\to 0}{\simeq} \sqrt{\frac{b_\Xi(1-b_\Xi)\hat v_H^2}{2\mu_\Xi^6}}$. One uses now the solution (4.17) to equate it to $\sqrt{\frac{b_X(1-b_X)\hat v_H^2}{2\mu_X^6}}$ which is indeed finite: the denominator is, up to mixing, given by the GMOR relation in terms of quark and meson masses (see (5.38) below and the remark next to it) and $\hat v_H^2$ is the VEV of the quasi-standard Higgs boson $\hat H^3$. The other possible solution of (4.17) would be $b_\Xi = 1$, but it does not correspond to a finite normalization for Ξ and we reject it.

Then, b_X, b_H, b_Ω are determined by (4.20) which yield, using (5.4):

$$\begin{aligned}
\frac{1}{1-b_\Omega} &= -1 + \delta\left(-\frac{t^2}{1-t^2}\left(\frac{1}{m^2_{\pi^+}} + \frac{1}{m^2_{D_s^+}}\right) + \frac{1}{1-t^2}\left(\frac{1}{m^2_{K^+}} + \frac{1}{m^2_{D^+}}\right)\right),\\
\frac{1}{1-b_X} - \frac{1}{1-b_H} &= \delta\left(\frac{1}{m^2_{\pi^+}} + \frac{1}{m^2_{K^+}} - \frac{1}{m^2_{D^+}} - \frac{1}{m^2_{D_s^+}}\right),\\
\frac{1}{1-b_X} + \frac{1}{1-b_H} &= \delta\left(\frac{1}{1-t^2}\left(\frac{1}{m^2_{\pi^+}} + \frac{1}{m^2_{D_s^+}}\right) - \frac{t^2}{1-t^2}\left(\frac{1}{m^2_{K^+}} + \frac{1}{m^2_{D^+}}\right)\right),
\end{aligned} \tag{5.5}$$

in which $t^2 \equiv \tan^2\theta_c$ is given by (5.1).

Calling

$$\begin{aligned}
r_1 &\equiv \frac{1}{m^2_{\pi^+}} + \frac{1}{m^2_{K^+}} + \frac{1}{m^2_{D^+}} + \frac{1}{m^2_{D_s^+}}, \quad r_2 \equiv \frac{1}{m^2_{\pi^+}} - \frac{1}{m^2_{K^+}} + \frac{1}{m^2_{D^+}} - \frac{1}{m^2_{D_s^+}},\\
r_3 &\equiv \frac{1}{m^2_{\pi^+}} + \frac{1}{m^2_{K^+}} - \frac{1}{m^2_{D^+}} - \frac{1}{m^2_{D_s^+}}, \quad r_4 \equiv \frac{1}{m^2_{\pi^+}} - \frac{1}{m^2_{K^+}} - \frac{1}{m^2_{D^+}} + \frac{1}{m^2_{D_s^+}},
\end{aligned} \tag{5.6}$$

one gets

$$\begin{aligned}
b_X &= 1 - \frac{2}{\delta\left(r_3 + \dfrac{1}{1-t^2}\dfrac{r_1+r_4}{2} - \dfrac{t^2}{1-t^2}\dfrac{r_1-r_4}{2}\right)} = 1 - \frac{2}{\delta\left(r_3 + \dfrac{r_1}{2} + \dfrac{r_4}{2\,c_{2d}}\right)},\\
b_H &= 1 - \frac{2}{\delta\left(-r_3 + \dfrac{1}{1-t^2}\dfrac{r_1+r_4}{2} - \dfrac{t^2}{1-t^2}\dfrac{r_1-r_4}{2}\right)} = 1 - \frac{2}{\delta\left(-r_3 + \dfrac{r_1}{2} + \dfrac{r_4}{2\,c_{2d}}\right)},\\
b_\Omega &= 1 - \frac{1}{-1 + \dfrac{\delta}{2}\left(r_1 - \dfrac{1+t^2}{1-t^2}\,r_4\right)} = 1 - \frac{1}{-1 + \dfrac{\delta}{2}\left(r_1 - \dfrac{r_4}{2\,c_{2d}}\right)}.
\end{aligned} \tag{5.7}$$

We plot in Fig.5.1 the 3 parameters b_X, b_H, b_Ω as functions of δ for the physical values (5.2) of the masses of the charged pseudoscalar mesons. Fig.5.1 already provides information on the value of δ because the b's, being the squared of ratios of VEV's supposedly real should be positive. We see that the strongest constraint is provided by the condition $b_H \geq 0$, which entails

$$\begin{aligned}
b_H \geq 0 \quad \Rightarrow \quad \delta &\geq \frac{m^2_{D^+}m^2_{D_s^+}\left(m^2_{K^+}m^2_{\pi^+}(m^2_{D_s^+} - m^2_{D^+}) + m^2_{D^+}m^2_{D_s^+}(m^2_{K^+} - m^2_{\pi^+})\right)}{m^2_{D^+}m^2_{D_s^+}(m^2_{K^+}+m^2_{D^+} - m^2_{D_s^+}m^2_{\pi^+}) + m^2_{K^+}m^2_{\pi^+}(m^4_{D_s^+} - m^4_{D^+})}\\
&= m^2_{D_s^+} + m^2_{\pi^+}\frac{m^2_{D_s^+}(m^2_{D_s^+} - m^2_{D^+})(m^2_{D^+} - m^2_{K^+})}{m^2_{K^+}m^4_{D^+}} + \mathcal{O}(m^4_{\pi^+}).
\end{aligned} \tag{5.8}$$

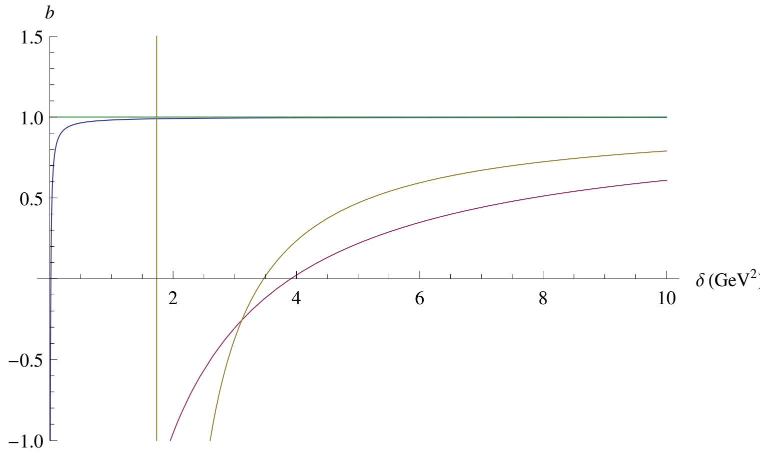

Figure 5.1: b_X (blue), b_H (purple) and b_Ω (yellow) as functions of δ at $\theta_u = 0$

Numerically, the first line of (5.8) yields

$$\delta \geq 3.90923\, GeV^2, \tag{5.9}$$

to be compared with $m_{D_s}^2 \approx 3.87495\, GeV^2$.

By (2.38) we accordingly know already that the quasi-standard Higgs boson $\hat{H}^3$ has a mass

$$m_{\hat{H}^3} \geq \sqrt{2} m_{D_s^+}, \tag{5.10}$$

which is to be compared with the case of 1 generation, for which the corresponding mass was $\sqrt{2} m_\pi$. As announced, this mass is controlled by the heaviest pseudoscalar meson (or quark) mass.

5.3 Neutral pseudoscalar mesons

5.3.1 Orthogonality relations

Eqs. (4.22) become at $\theta_u = 0$

$$
\begin{aligned}
&(a)\ D^0 \perp K^0 : c_{2d}(-\frac{\delta_\Omega}{\nu_\Omega^4} + \frac{\hat\delta_\Omega}{\hat\nu_\Omega^4}) - (\frac{\delta_\Xi}{\nu_\Xi^4} - \frac{\hat\delta_\Xi}{\hat\nu_\Xi^4}) = 0,\\
&(b)\ \bar D^0 \perp \bar K^0 : idem,\\
&(c)\ D^0 \perp \bar K^0 : c_{2d}(-\frac{\delta_\Omega}{\nu_\Omega^4} + \frac{\hat\delta_\Omega}{\hat\nu_\Omega^4}) + (\frac{\delta_\Xi}{\nu_\Xi^4} - \frac{\hat\delta_\Xi}{\hat\nu_\Xi^4}) = 0,\\
&(d)\ \bar D^0 \perp K^0 : idem,\\
&(e)\ \pi^0 \perp (D^0 + \bar D^0) : -s_{2d}\frac{\delta_\Omega}{\nu_\Omega^4} + s_{2d}\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4} = 0,\\
&(f)\ \eta \perp (D^0 + \bar D^0) : s_{2d}\frac{\delta_\Omega}{\nu_\Omega^4} - s_{2d}\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4} = 0,\\
&\pi^0 \perp (K^0 - \bar K^0) : always\ true,\\
&\eta \perp (K^0 - \bar K^0) : always\ true,\\
&\pi^0 \perp (D^0 - \bar D^0) : always\ true,\\
&\eta \perp (D^0 - \bar D^0) : always\ true,\\
&(K^0 + \bar K^0) \perp (K^0 - \bar K^0) : always\ true,\\
&(D^0 + \bar D^0) \perp (D^0 - \bar D^0) : always\ true,\\
&(g)\ \pi^0 \perp (K^0 + \bar K^0) : s_d c_d \left(-(1+c_d^2)\frac{\delta_X}{\nu_X^4} + (1-c_d^2)\frac{\hat\delta_X}{\hat\nu_X^4} + s_d^2\frac{\delta_H}{\nu_H^4} + s_d^2\frac{\hat\delta_H}{\hat\nu_H^4}\right) + \frac{c_{2d}s_{2d}}{2}\left(\frac{\delta_\Omega}{\nu_\Omega^4} + \frac{\hat\delta_\Omega}{\hat\nu_\Omega^4}\right) = 0,\\
&(h)\ \eta \perp (K^0 + \bar K^0) : s_d c_d \left(-(1-c_d^2)\frac{\delta_X}{\nu_X^4} + (1+c_d^2)\frac{\hat\delta_X}{\hat\nu_X^4} - s_d^2\frac{\delta_H}{\nu_H^4} - s_d^2\frac{\hat\delta_H}{\hat\nu_H^4}\right) - \frac{c_{2d}s_{2d}}{2}\left(\frac{\delta_\Omega}{\nu_\Omega^4} + \frac{\hat\delta_\Omega}{\hat\nu_\Omega^4}\right) = 0,\\
&(i)\ D^0 \perp \bar D^0 : always\ true,\\
&(j)\ K^0 \perp \bar K^0 : s_d^2 c_d^2(\frac{\delta_X}{\nu_X^4} + \frac{\hat\delta_X}{\hat\nu_X^4} + \frac{\delta_H}{\nu_H^4} + \frac{\hat\delta_H}{\hat\nu_H^4}) + \frac{1}{2}c_{2d}^2(\frac{\delta_\Omega}{\nu_\Omega^4} + \frac{\hat\delta_\Omega}{\hat\nu_\Omega^4}) - \frac{1}{2}(\frac{\delta_\Xi}{\nu_\Xi^4} + \frac{\hat\delta_\Xi}{\hat\nu_\Xi^4}) = 0.
\end{aligned}
\tag{5.11}
$$

• (a) and (c) of (5.11) yield

$$\frac{\delta_\Omega}{\nu_\Omega^4} = \frac{\hat\delta_\Omega}{\hat\nu_\Omega^4}, \quad \frac{\delta_\Xi}{\nu_\Xi^4} = \frac{\hat\delta_\Xi}{\hat\nu_\Xi^4}, \tag{5.12}$$

which also makes (e) and (f) of (5.11) true.

Equations (4.16), (5.12) summarize into

$$\frac{1-b_X}{\nu_X^4} = \frac{1-b_H}{\nu_H^4} = \frac{1-b_\Omega}{\nu_\Omega^4} = \frac{1-b_\Xi}{\nu_\Xi^4} = \frac{1-\hat b_\Omega}{\hat\nu_\Omega^4} = \frac{1-\hat b_\Xi}{\hat\nu_\Xi^4}, \tag{5.13}$$

which is equivalent to

$$\frac{b_X(1-b_X)}{\mu_X^6} = \frac{b_H(1-b_H)}{\mu_H^6} = \frac{b_\Omega(1-b_\Omega)}{\mu_\Omega^6} = \frac{b_\Xi(1-b_\Xi)}{\mu_\Xi^6} = \frac{\hat b_\Omega(1-\hat b_\Omega)}{\hat\mu_\Omega^6} = \frac{\hat b_\Xi(1-\hat b_\Xi)}{\hat\mu_\Xi^6}. \tag{5.14}$$

By the definition (4.8), this also translates into

$$
\begin{aligned}
&r_H = \sqrt{\frac{b_H(1-b_H)}{b_X(1-b_X)}}, \quad r_\Omega = \sqrt{\frac{b_\Omega(1-b_\Omega)}{b_X(1-b_X)}}, \quad r_\Xi = \sqrt{\frac{b_\Xi(1-b_\Xi)}{b_X(1-b_X)}},\\
&\hat r_\Omega = \sqrt{\frac{\hat b_\Omega(1-\hat b_\Omega)}{b_X(1-b_X)}}, \quad \hat r_\Xi = \sqrt{\frac{\hat b_\Xi(1-\hat b_\Xi)}{b_X(1-b_X)}},
\end{aligned}
\tag{5.15}
$$

to which should of course be added the definition / statements

$$\hat{b}_H = 1, \quad r_X = 1, \quad \mu_\Xi^3 = 0 = \hat{\mu}_\Xi^3 \quad \Rightarrow \quad r_\Xi = 0 = \hat{r}_\Xi. \tag{5.16}$$

Eqs. (5.14) establish relations between bosonic and fermionic VEV's, therefore between gauge and chiral symmetry breaking.

• Mass-like $K^0\bar{K}^0$ non-diagonal terms are proportional to

$$\frac{s_d^2 c_d^2}{2}\left(\frac{\delta_X}{\nu_X^4} + \frac{\hat{\delta}_X}{\hat{\nu}_X^4} + \frac{\delta_H}{\nu_H^4} + \overbrace{\frac{\hat{\delta}_H}{\hat{\nu}_H^4}}^{0}\right) + \frac{c_{2d}^2}{4}\left(\frac{\delta_\Omega}{\nu_\Omega^4} + \frac{\hat{\delta}_\Omega}{\hat{\nu}_\Omega^4}\right) - \frac{1}{4}\left(\frac{\delta_\Xi}{\nu_\Xi^4} + \frac{\hat{\delta}_\Xi}{\hat{\nu}_\Xi^4}\right). \tag{5.17}$$

From the conditions (5.12) for the absence of neutral $K - D$ transitions the last two terms become $\frac{c_{2d}^2}{2}\frac{\delta_\Omega}{\nu_\Omega^4} - \frac{1}{2}\frac{\delta_\Xi}{\nu_\Xi^4}$; then, from (4.16) for charged mesons, they become $-\frac{s_{2d}^2}{2}\frac{\delta_\Xi}{\nu_\Xi^4}$, which is also $-\frac{s_{2d}^2}{2}\frac{\delta_X}{\nu_X^4}$; the first term transform by (4.16) into $\frac{s_d^2 c_d^2}{2}(2\frac{\delta_X}{\nu_X^4} + \frac{\hat{\delta}_X}{\hat{\nu}_X^4})$ such that the $K^0\bar{K}^0$ terms are finally proportional to $s_d^2 c_d^2\left(\frac{1}{2}\frac{\hat{\delta}_X}{\hat{\nu}_X^4} - \frac{\delta_X}{\nu_X^4}\right)$. That they vanish requires accordingly

$$\frac{\delta_X}{\nu_X^4} = \frac{1}{2}\frac{\hat{\delta}_X}{\hat{\nu}_X^4}. \tag{5.18}$$

Using (2.31) and the definitions (4.9), (5.18) is equivalent to

$$\frac{1 - b_X}{\nu_X^4} = \frac{1}{2}\frac{1 - \hat{b}_X}{\hat{\nu}_X^4} \Leftrightarrow \frac{b_X(1 - b_X)}{\mu_X^6} = \frac{1}{2}\frac{\hat{b}_X(1 - \hat{b}_X)}{\hat{\mu}_X^6}. \tag{5.19}$$

• Using $\hat{\delta}_H = 0$ (2.35) and (4.17) for $X, H, \Omega, \hat{\Omega}$, (7) and (8) of (5.11) transform respectively into

$$\frac{\delta_X}{\nu_X^4} = \frac{1}{2}\frac{\hat{\delta}_X}{\hat{\nu}_X^4}, \tag{5.20}$$

and

$$\frac{\delta_X}{\nu_X^4} = \frac{1 + c_d^2}{2c_d^2}\frac{\hat{\delta}_X}{\hat{\nu}_X^4}. \tag{5.21}$$

(5.20) and (5.21) are incompatible. (5.20) is the same condition as (5.18) that we found for canceling non-diagonal $K^0 - \bar{K}^0$ couplings. So, while one can easily achieve the orthogonality of π^0 to both $K^0 \pm \bar{K}^0$, this is not the case for η.

• For the same reasons as we chose $b_\Xi = 0$ (see (5.4)), (5.13) and the requirement that the normalization factor of the $\hat{\Xi}$ quadruplet be finite allows to take

$$\boxed{\hat{b}_\Xi = 0} \tag{5.22}$$

Relations (5.4) and (5.22) largely simplify the calculations [1].

5.3.2 Masses of K^0 and D^0

One gets from (4.24) and (4.25)

$$m_{K^0}^2 = \frac{\frac{s_d^2 c_d^2}{2}\left(\frac{\delta_X}{\nu_X^4} + \frac{\hat{\delta}_X}{\hat{\nu}_X^4} + \frac{\delta_H}{\nu_H^4} + \frac{\hat{\delta}_H}{\hat{\nu}_H^4}\right) + \frac{c_{2d}^2}{4}\left(\frac{\delta_\Omega}{\nu_\Omega^4} + \frac{\hat{\delta}_\Omega}{\hat{\nu}_\Omega^4}\right) + \frac{1}{4}\left(\frac{\delta_\Xi}{\nu_\Xi^4} + \frac{\hat{\delta}_\Xi}{\hat{\nu}_\Xi^4}\right)}{\frac{s_d^2 c_d^2}{2}\left(\frac{1}{\nu_X^4} + \frac{1}{\hat{\nu}_X^4} + \frac{1}{\nu_H^4} + \frac{1}{\hat{\nu}_H^4}\right) + \frac{c_{2d}^2}{4}\left(\frac{1}{\nu_\Omega^4} + \frac{1}{\hat{\nu}_\Omega^4}\right) + \frac{1}{4}\left(\frac{1}{\nu_\Xi^4} + \frac{1}{\hat{\nu}_\Xi^4}\right)}, \tag{5.23}$$

and

$$m_{D^0}^2 = \frac{\frac{\delta_\Omega}{\nu_\Omega^4} + \frac{\hat{\delta}_\Omega}{\hat{\nu}_\Omega^4} + \frac{\delta_\Xi}{\nu_\Xi^4} + \frac{\hat{\delta}_\Xi}{\hat{\nu}_\Xi^4}}{\frac{1}{\nu_\Omega^4} + \frac{1}{\hat{\nu}_\Omega^4} + \frac{1}{\nu_\Xi^4} + \frac{1}{\hat{\nu}_\Xi^4}}. \tag{5.24}$$

[1] They should be relaxed for 3 generations since, at least at the perturbative (2-loops) level, $< \bar{c}u - \bar{u}c >$ and $< \bar{d}s - \bar{s}d >$ get proportional to the CP-violating phase of the Cabibbo-Kobayashi-Maskawa mixing matrix.

Using (2.31), (5.13) (which entails $\hat{\delta}_H = 0$), (5.4) and (5.22) (which cancel the terms in δ_Ξ and $\hat{\delta}_\Xi$ in their numerators), (5.23) and (5.24) rewrite

$$m^2_{K^0} = \frac{4\delta}{\frac{1}{2}s^2_{2d}\left(\frac{1}{1-b_X}+\frac{2}{1-\hat{b}_X}+\frac{1}{1-b_H}+\frac{1}{b_X(1-b_X)}\frac{1}{\hat{r}^2_H}\right)+c^2_{2d}\left(\frac{1}{1-b_\Omega}+\frac{1}{1-\hat{b}_\Omega}\right)+2}, \tag{5.25}$$

and

$$m^2_{D_0} = \frac{4\delta}{\frac{1}{1-b_\Omega}+\frac{1}{1-\hat{b}_\Omega}+2}. \tag{5.26}$$

(5.25) and (5.26) combine into

$$\frac{m^2_{K_0}}{m^2_{D_0}} = \frac{\frac{1}{1-b_\Omega}+\frac{1}{1-\hat{b}_\Omega}+2}{\frac{1}{2}s^2_{2d}\left(\frac{1}{1-b_X}+\frac{2}{1-\hat{b}_X}+\frac{1}{1-b_H}+\frac{1}{b_X(1-b_X)}\frac{1}{\hat{r}^2_H}\right)+c^2_{2d}\left(\frac{1}{1-b_\Omega}+\frac{1}{1-\hat{b}_\Omega}\right)+2}. \tag{5.27}$$

Since b_Ω is known by (5.5) as a function of δ, (4.25) defines $\hat{b}_\Omega$ as a function of δ, too.

Eq. (5.27) involves $\hat{r}^2_H \equiv \hat{\mu}^6_H/\mu^6_X$, $\sin^2 2\theta_d \equiv 4t^2/(1+t^2)^2$ and $\cos^2 2\theta_d \equiv \left(\frac{1-t^2}{1+t^2}\right)^2$ are known by (5.1), b_X, b_H, b_Ω and $\hat{b}_\Omega$ are known as functions of δ; therefore, (5.27) defines $\hat{b}_X$ as a function of δ and $\hat{r}_H$.

We avoid at the moment to use $\pi^0, \eta \ldots$ as inputs. This is because these mesons are known to mix and, accordingly, there is uncertainty concerning their interpolating fields in terms of quark bilinears. We shall see later that the π^0 as we defined it, proportional to $\bar{u}\gamma_5 u - \bar{d}\gamma_5 d$, gets a suitable mass (together with appropriate orthogonality relations as we already mentioned in subsection 5.3.1).

5.3.3 First hints at a problem concerning $\hat{b}_X$

One can already get from (5.27) some valuable information concerning $\hat{b}_X \equiv \left(\frac{\hat{v}_X}{\hat{v}_H}\right)^2 = \left(\frac{<\hat{X}^3>}{<\hat{H}^3>}\right)^2$. One has to make a reasonable estimate of $\hat{r}^2_H$. This is fairly easy from its definition since $\hat{r}_H \equiv \frac{\hat{\mu}^3_H}{\mu^3_X} = \frac{<\bar{c}c-\bar{s}s>}{<\bar{u}u+\bar{d}d>}$: its modulus is presumably ≤ 1 because heavy quarks being "more classical" that light quarks should undergo less condensation in the vacuum. We shall see later from fermionic considerations that, indeed $\hat{r}_H \approx .6$. Numerical evaluations then show that, for $\delta \geq m^2_{D_s}$,

$$\hat{b}_X > 1. \tag{5.28}$$

This is confirmed by a formal expansion at the chiral limit $m_\pi \to 0$ [2]

$$\hat{b}_X \overset{m_\pi \to 0}{\simeq} 1 + \frac{m^2_\pi}{\delta}\frac{2\hat{r}^2_H}{1-\hat{r}^2_H} + \mathcal{O}(m^4_\pi) > 1. \tag{5.29}$$

A similar expansion for b_X is

$$b_X \overset{m_\pi \to 0}{\simeq} 1 - \frac{m^2_\pi}{\delta} + \mathcal{O}(m^4_\pi) < 1, \tag{5.30}$$

which is in fair agreement with the curve on Fig.5.1.

Eq. (5.20), which controls the orthogonality of π^0 to $K^0 + \bar{K}^0$, identical to (5.18) which controls the orthogonality of K^0 to $\bar{K}^0$ are incompatible with (5.29) and (5.30): indeed, $b_X < 1$ and $\hat{b}_X > 1$ lead, by (5.20), to $\hat{\nu}^4_X/\nu^4_X \equiv (b_X/\hat{b}_X)(\hat{\mu}^3_X/\mu^3_X)^2 < 0$. Since b_X and $\hat{b}_X$ are real, this could only occur for $\hat{\mu}^3_X/\mu^3_X \equiv <\bar{u}u - \bar{d}d> / <\bar{u}u + \bar{d}d>$ imaginary. μ^3_X being real by the GMOR relation (see (5.38) below), $\hat{\mu}^3_X \equiv <\bar{u}u - \bar{d}d> /\sqrt{2}$ should be imaginary. This is manifestly impossible since $(\bar{q}_i q_i)^\dagger = \bar{q}_i q_i$.

[2] Such expansions cannot always be trusted. However, numerical checks show that (5.29) is reasonably accurate. In particular, the pole at $\hat{r}^2_H = 1$ of (5.29) only gets moved in the exact formula to $\hat{r}^2_H \approx 1.08$.

5.4 Charged scalars

The same investigation that we did in section 5.1 for charged pseudoscalars we now do for charged scalars. It is not our goal here to extensively study the spectrum of charged scalar mesons. We want only to show that they are expected to align with flavor eigenstates. Like for charged pseudoscalars, we shall consider a priori that the interpolating field of the charged scalar pion π^{s+} is proportional to $\bar{u}_m d_m$, that of the charged scalar D_s meson, that we note D_s^{s+}, is proportional to $\bar{c}_m s_m$ *etc*.

5.4.1 Orthogonality relations

The set of equations equivalent to (4.13) is (remember that we work at $\theta_u = 0$)

$$\begin{aligned}
\frac{s_d c_d}{\overline{\hat{\nu}_X}^4} - \frac{1}{2}\frac{s_d c_d}{\overline{\hat{\nu}_\Omega}^4} - \frac{1}{2}\frac{s_d c_d}{\overline{\hat{\nu}_\Xi}^4} &= 0 \qquad (\pi^{s+} \not\leftrightarrow K^{s+}),\\
-\frac{1}{2}\frac{s_d c_d}{\overline{\hat{\nu}_\Omega}^4} + \frac{1}{2}\frac{s_d c_d}{\overline{\hat{\nu}_\Xi}^4} &= 0 \qquad (\pi^{s+} \not\leftrightarrow D^{s+}),\\
-\frac{1}{2}\frac{s_d^2}{\overline{\hat{\nu}_\Omega}^4} + \frac{1}{2}\frac{s_d^2}{\overline{\hat{\nu}_\Xi}^4} &= 0 \qquad (\pi^{s+} \not\leftrightarrow D_s^{s+}),\\
+\frac{1}{2}\frac{c_d^2}{\overline{\hat{\nu}_\Omega}^4} - \frac{1}{2}\frac{c_d^2}{\overline{\hat{\nu}_\Xi}^4} &= 0 \qquad (K^{s+} \not\leftrightarrow D^{s+}),\\
+\frac{1}{2}\frac{s_d c_d}{\overline{\hat{\nu}_\Omega}^4} - \frac{1}{2}\frac{s_d c_d}{\overline{\hat{\nu}_\Xi}^4} &= 0 \qquad (K^{s+} \not\leftrightarrow D_s^{s+}),\\
\frac{s_d c_d}{\overline{\hat{\nu}_H}^4} + \frac{1}{2}\frac{s_d c_d}{\overline{\hat{\nu}_\Omega}^4} + \frac{1}{2}\frac{s_d c_d}{\overline{\hat{\nu}_\Xi}^4} &= 0 \qquad (D^{s+} \not\leftrightarrow D_s^{s+}).
\end{aligned} \tag{5.31}$$

The main difference between (5.31) and (4.13) is that $\hat{\delta}_H = \delta(1-\hat{b}_H) = 0$ while δ_H is different from 0. One term is consequently "missing" in equation 6 of (5.31).

Equations 2 to 5 of (5.31) lead to $\frac{1-\hat{b}_\Omega}{\hat{\nu}_\Omega^4} = \frac{1-\hat{b}_\Xi}{\hat{\nu}_\Xi^4}$, which has already been obtained from pseudoscalar mesons in (5.13). Then, equation 1 leads to

$$\frac{1-\hat{b}_X}{\hat{\nu}_X^4} = \frac{1-\hat{b}_\Omega}{\hat{\nu}_\Omega^4}, \tag{5.32}$$

while, because $\hat{b}_H = 1$, equation 6 entails

$$\frac{1-\hat{b}_X}{\hat{\nu}_X^4} = -\frac{1-\hat{b}_\Omega}{\hat{\nu}_\Omega^4}. \tag{5.33}$$

Both (5.32) and (5.33) are in contradiction with (5.20). (5.32) is incompatible because it does not exhibit the factor $1/2$ present in (5.20) (we recall that by (5.13) $\frac{1-\hat{b}_\Omega}{\hat{\nu}_\Omega^4} = \frac{1-b_X}{\nu_X^4}$); (5.33) is also manifestly incompatible with (5.20).

The only way to reconcile these orthogonality relations with the results that we have obtained for pseudoscalar mesons is to turn to 0 the mixing angle for scalars, that is to set $\theta_d = 0$ in (5.31). Then, scalar mesons are bond states of quark flavor eigenstates.

5.4.2 Masses of charged scalars

Let us confirm this proposition by focusing on the (π^{s+}, K^{s+}) and (D^{s+}, D_s^{s+}) systems. They indeed correspond to the problematic equations 1 and 6 in the system (5.31).

As far as the second pair is concerned, one finds that its mass matrix is proportional to

$$\delta \begin{pmatrix} c_d^2 & s_d c_d \\ s_d c_d & s_d^2 \end{pmatrix}, \tag{5.34}$$

which displays, as expected, a vanishing eigenvalue : the model has been indeed built such that the *flavor* $D_s^{s\pm}$ are the true Goldstones of the broken $SU(2)_L$.

We then focus on the (π^{s+}, K^{s+}) system. Its mass matrix is found to be proportional to

$$\delta \left(\begin{array}{cc} \frac{c_d^2}{\hat{\nu}_X^4}(1-\hat{b}_X) + \frac{s_d^2}{\hat{\nu}_\Omega^4}(1-\hat{b}_\Omega) & -\frac{s_d c_d}{\hat{\nu}_\Omega^4}(1-\hat{b}_\Omega) \\ -\frac{s_d c_d}{\hat{\nu}_\Omega^4}(1-\hat{b}_\Omega) & \frac{s_d^2}{\hat{\nu}_X^4}(1-\hat{b}_X) + \frac{c_d^2}{\hat{\nu}_\Omega^4}(1-\hat{b}_\Omega) \end{array}\right) \propto \delta \left(\begin{array}{cc} s_d^2 - \epsilon & -s_d c_d \\ -s_d c_d & c_d^2 - \epsilon, \end{array}\right), \tag{5.35}$$

in which ϵ is a small positive number. (5.35) displays a small negative eigenvalue $-\epsilon$. Since the corresponding kinetic terms, proportional to

$$\left(\begin{array}{cc} \frac{c_d^2}{\hat{\nu}_X^4} + \frac{s_d^2}{\hat{\nu}_\Omega^4} & -\frac{1}{2}\frac{s_d c_d}{\hat{\nu}_\Omega^4} - \frac{1}{2}\frac{s_d c_d}{\hat{\nu}_\Xi^4} \\ -\frac{1}{2}\frac{s_d c_d}{\hat{\nu}_\Omega^4} - \frac{1}{2}\frac{s_d c_d}{\hat{\nu}_\Xi^4} & \frac{s_d^2}{\hat{\nu}_X^4} + \frac{c_d^2}{\hat{\nu}_\Omega^4} \end{array}\right), \tag{5.36}$$

have 2 positive eigenvalues, we face the issue that, for 2 generations, the binary system of charged scalar pions and kaons, if aligned with quark mass eigenstates, involves a tachyonic state.

This problem fades away when setting, like in the previous subsection, $\theta_d = 0$, that is aligning scalar mesons with flavor eigenstates. The masses of the scalar charged pion and kaon become, then, proportional to $\frac{1-\hat{b}_X}{\hat{\nu}_X^4}$ and $\frac{1-\hat{b}_\Omega}{\hat{\nu}_\Omega^4}$ that is, by (5.20) and (5.14), to $2\frac{1-b_X}{\nu_X^4} = 2\frac{b_X(1-b_X)}{\mu_X^6}$ and $\frac{1-b_X}{\nu_X^4} = \frac{b_X(1-b_X)}{\mu_X^6}$.

We therefore conclude that the alignment of the 3 Goldstone bosons of the broken gauge $SU(2)_L$ symmetry with flavor eigenstates triggers the same alignment for charged scalar mesons.

5.5 Summary of bosonic constraints

We summarize below the results that we have obtained from the sole bosonic constraints and list what remains to be done to determine all the parameters of the theory.

We need the reference points for the b's and the r's, that is $\hat{v}_H$ and μ_X^3. μ_X^3 is a condensate of quark flavor eigenstates while the low energy theorems (PCAC, GMOR relation) involve mass eigenstates:

$$i(m_u + m_d)\bar{u}_m \gamma_5 d_m = \sqrt{2} f_\pi m_{\pi^+}^2 \pi^+, \tag{5.37}$$

$$(m_u + m_d) < \bar{u}_m u_m + \bar{d}_m d_m >= 2 f_\pi^2 m_{\pi^+}^2. \tag{5.38}$$

Since the mixing angle θ_c is small, we shall make the approximation that μ_X^3 is close to the quark condensate of mass eigenstates $\mu_X^3 \approx \frac{<\bar{u}_m u_m + \bar{d}_m d_m>}{\sqrt{2}}$ which is given by the GMOR relation (5.38). We shall accordingly approximate

$$\mu_X^3 \equiv \frac{< \bar{u}u + \bar{d}d >}{\sqrt{2}} \approx \frac{< \bar{u}_m u_m + \bar{d}_m d_m >}{\sqrt{2}} = \frac{\sqrt{2} f_\pi^2 m_\pi^2}{m_u + m_d}. \tag{5.39}$$

By (5.5) and (5.24), $b_X, b_H, b_\Omega, \hat{b}_\Omega$ are known functions of δ; therefore, by (5.15), $r_H, r_\Omega, \hat{r}_\Omega$ are also known functions of δ. By (5.27), $\hat{b}_X$ is a known function of δ and $\hat{r}_H$.

Numerically, one gets (δ being expressed in GeV^2)

$$b_X \equiv \left(\frac{v_X}{\hat{v}_H}\right)^2 \approx 1 - \frac{0.0181324}{\delta}, \quad b_H \equiv \left(\frac{v_H}{\hat{v}_H}\right)^2 \approx 1 - \frac{3.90923}{\delta}, \quad b_\Omega \equiv \left(\frac{v_\Omega}{\hat{v}_H}\right)^2 \approx 1 - \frac{1}{-1 + 0.576693\,\delta},$$
$$\hat{b}_X \equiv \left(\frac{\hat{v}_X}{\hat{v}_H}\right)^2 \approx \frac{0.0017312\,\hat{r}_H^2 - 0.113608\,\delta\,\hat{r}_H^2 + \delta^2(-0.87758 + \hat{r}_H^2)}{0.00115414\,\hat{r}_H^2 - 0.0817829\,\delta\,\hat{r}_H^2 + \delta^2(-0.87758 + \hat{r}_H^2)}, \quad \hat{b}_\Omega \equiv \left(\frac{\hat{v}_\Omega}{\hat{v}_H}\right)^2 \approx 1 - \frac{1}{-1 + 0.573491\,\delta}, \tag{5.40}$$

to which should be added $\hat{b}_H = 1$ by definition and $b_\Xi = 0 = \hat{b}_\Xi$ (5.4) and (5.22).

As will be confirmed later, δ stands very close to $m^2_{D_s}$, such that we already know that b_X is very close to and smaller that 1, b_H is very small, $b_\Omega \approx \hat b_\Omega \approx .2$. Since $\hat r^2_H$ is of order 1, $\hat b_X$ is very close to 1 and slightly larger. This gives already interesting results concerning the Higgs spectrum which varies like $\sqrt{b}$ (see (2.39)): $X^0, \hat X^3, \hat H^3$ are quasi degenerate, H^0 is very light, Ω^0 and $\hat\Omega^3$ have intermediate mass scales. As far as Ξ^0 and $\hat\Xi^3$ are concerned, they are at the moment massless but they cannot be true Goldstones bosons and we shall show in subsection 5.10.3 that they are expected to get small masses by quantum corrections.

Hierarchies between bosonic VEV's are accordingly $\mathcal{O}(1)$ except for $\hat v_H/v_H \equiv 1/\sqrt{b_H}$ which is still large, but much smaller than for 1 generation (see also subsection 5.10.2). This is the same type of "see-saw" mechanism that we witnessed for one generation between the quasi-standard Higgs doublet and its parity transformed, and which gives birth to a very light Higgs bosons H^0.

Equation (4.11) determines the value of $\hat v_H$ as a function of δ and $\hat r_H$

$$\boxed{\hat v_H = \frac{2m_W}{g}\frac{1}{\sqrt{b_X + b_H + b_\Omega + 1 + \hat b_X + \hat b_\Omega}}} \tag{5.41}$$

Since, as we shall confirm, $\delta \approx m^2_{D_s}$, $b_X \approx 1 \approx \hat b_X$, $b_\Omega \approx \hat b_\Omega \approx .2$, one can already state

$$\hat v_H \approx 143\ GeV. \tag{5.42}$$

The value of $\hat v_H$ comes out smaller than in the GSW model because the W mass receives contributions from the VEV's of several Higgs bosons. Each of them has therefore less to contribute.

Once the b's are determined, (2.31) gives the values of the δ_i's and (5.15) provides the ratios of fermionic VEV's (except $\hat r_X$ which is problematic, see section 5.11 below). We shall come back later to their numerical values.

Accordingly, at this point, δ and $\hat r_H$ are still to be determined, together with the $\delta_{i\hat i} = -\kappa_{\hat i i}$'s. This makes a total of 10 parameters still to be determined to have full control of the theory. For what concerns us here, mainly the spectrum of Higgs bosons, we mainly need δ and $\hat v_H$. The other parameters will only be needed to determine the couplings of the various fields to each other.

In section 5.6, we shall use the fermionic sector of the theory to determine δ and $\hat v_H$.

5.6 General fermionic constraints

Yukawa couplings provide quark mass terms as functions of the various parameters and VEV's. Setting θ_u to 0 constrains in particular the non-diagonal μ_{uc} and μ_{cu} mass terms to vanish.

5.6.1 Quark mass terms

From the Yukawa Lagrangian (2.23) and $i \in [X, H, \Omega, \Xi]$, one gets the following diagonal quark mass terms

$$
\begin{aligned}
\mu_u &= \frac{\hat{v}_H^2}{2\sqrt{2}} \left[\frac{\delta(1-b_X)b_X}{\mu_X^3} + \frac{\delta(1-\hat{b}_X)\hat{b}_X}{\hat{\mu}_X^3} + \delta_{X\hat{X}} \sqrt{b_X \hat{b}_X} \left(\frac{1}{\mu_X^3} - \frac{1}{\hat{\mu}_X^3} \right) \right], \\
\mu_d &= \frac{\hat{v}_H^2}{2\sqrt{2}} \left[\frac{\delta(1-b_X)b_X}{\mu_X^3} - \frac{\delta(1-\hat{b}_X)\hat{b}_X}{\hat{\mu}_X^3} + \delta_{X\hat{X}} \sqrt{b_X \hat{b}_X} \left(\frac{1}{\mu_X^3} + \frac{1}{\hat{\mu}_X^3} \right) \right], \\
\mu_c &= \frac{\hat{v}_H^2}{2\sqrt{2}} \left[\frac{\delta(1-b_H)b_H}{\mu_H^3} + \frac{\delta\overbrace{(1-\hat{b}_H)}^{0}\hat{b}_H}{\hat{\mu}_H^3} + \delta_{H\hat{H}} \sqrt{b_H \hat{b}_H} \left(\frac{1}{\mu_H^3} - \frac{1}{\hat{\mu}_H^3} \right) \right] \\
&= \frac{\hat{v}_H^2}{2\sqrt{2}} \left[\frac{\delta(1-b_H)b_H}{\mu_H^3} + \delta_{H\hat{H}} \sqrt{b_H} \left(\frac{1}{\mu_H^3} - \frac{1}{\hat{\mu}_H^3} \right) \right], \\
\mu_s &= \frac{\hat{v}_H^2}{2\sqrt{2}} \left[\frac{\delta(1-b_H)b_H}{\mu_H^3} - \frac{\delta\overbrace{(1-\hat{b}_H)}^{0}\hat{b}_H}{\hat{\mu}_H^3} + \delta_{H\hat{H}} \sqrt{b_H \hat{b}_H} \left(\frac{1}{\mu_H^3} + \frac{1}{\hat{\mu}_H^3} \right) \right] \\
&= \frac{\hat{v}_H^2}{2\sqrt{2}} \left[\frac{\delta(1-b_H)b_H}{\mu_H^3} + \delta_{H\hat{H}} \sqrt{b_H} \left(\frac{1}{\mu_H^3} + \frac{1}{\hat{\mu}_H^3} \right) \right],
\end{aligned}
\tag{5.43}
$$

and the following non-diagonal quark mass terms

$$
\begin{aligned}
\mu_{uc} &= \frac{\hat{v}_H^2}{4} \left[\frac{\delta(1-b_\Omega)b_\Omega}{\mu_\Omega^3} + \frac{\delta(1-\hat{b}_\Omega)\hat{b}_\Omega}{\hat{\mu}_\Omega^3} + \delta_{\Omega\hat{\Omega}} \sqrt{b_\Omega \hat{b}_\Omega} \left(\frac{1}{\mu_\Omega^3} - \frac{1}{\hat{\mu}_\Omega^3} \right) \right. \\
&\qquad \left. + \frac{\delta(1-b_\Xi)b_\Xi}{\mu_\Xi^3} + \frac{\delta(1-\hat{b}_\Xi)\hat{b}_\Xi}{\hat{\mu}_\Xi^3} + \delta_{\Xi\hat{\Xi}} \sqrt{b_\Xi \hat{b}_\Xi} \left(\frac{1}{\mu_\Xi^3} - \frac{1}{\hat{\mu}_\Xi^3} \right) \right] \leftarrow = 0 \\
&= \frac{\hat{v}_H^2}{4} \left[\frac{\delta(1-b_\Omega)b_\Omega}{\mu_\Omega^3} + \frac{\delta(1-\hat{b}_\Omega)\hat{b}_\Omega}{\hat{\mu}_\Omega^3} + \delta_{\Omega\hat{\Omega}} \sqrt{b_\Omega \hat{b}_\Omega} \left(\frac{1}{\mu_\Omega^3} - \frac{1}{\hat{\mu}_\Omega^3} \right) \right] \\
\mu_{cu} &= \frac{\hat{v}_H^2}{4} \left[\frac{\delta(1-b_\Omega)b_\Omega}{\mu_\Omega^3} + \frac{\delta(1-\hat{b}_\Omega)\hat{b}_\Omega}{\hat{\mu}_\Omega^3} + \delta_{\Omega\hat{\Omega}} \sqrt{b_\Omega \hat{b}_\Omega} \left(\frac{1}{\mu_\Omega^3} - \frac{1}{\hat{\mu}_\Omega^3} \right) \right. \\
&\qquad \left. - \frac{\delta(1-b_\Xi)b_\Xi}{\mu_\Xi^3} - \frac{\delta(1-\hat{b}_\Xi)\hat{b}_\Xi}{\hat{\mu}_\Xi^3} - \delta_{\Xi\hat{\Xi}} \sqrt{b_\Xi \hat{b}_\Xi} \left(\frac{1}{\mu_\Xi^3} - \frac{1}{\hat{\mu}_\Xi^3} \right) \right] \leftarrow = 0 \\
&= \frac{\hat{v}_H^2}{4} \left[\frac{\delta(1-b_\Omega)b_\Omega}{\mu_\Omega^3} + \frac{\delta(1-\hat{b}_\Omega)\hat{b}_\Omega}{\hat{\mu}_\Omega^3} + \delta_{\Omega\hat{\Omega}} \sqrt{b_\Omega \hat{b}_\Omega} \left(\frac{1}{\mu_\Omega^3} - \frac{1}{\hat{\mu}_\Omega^3} \right) \right]
\end{aligned}
\tag{5.44}
$$

$$
\begin{aligned}
\mu_{ds} &= \frac{\hat v_H^2}{4}\left[\frac{\delta(1-b_\Omega)b_\Omega}{\mu_\Omega^3} - \frac{\delta(1-\hat b_\Omega)\hat b_\Omega}{\hat\mu_\Omega^3} + \delta_{\Omega\hat\Omega}\sqrt{b_\Omega\hat b_\Omega}\left(\frac{1}{\mu_\Omega^3}+\frac{1}{\hat\mu_\Omega^3}\right)\right.\\
&\qquad \left. +\frac{\delta(1-b_\Xi)b_\Xi}{\mu_\Xi^3} - \frac{\delta(1-\hat b_\Xi)\hat b_\Xi}{\hat\mu_\Xi^3} + \delta_{\Xi\hat\Xi}\sqrt{b_\Xi\hat b_\Xi}\left(\frac{1}{\mu_\Xi^3}+\frac{1}{\hat\mu_\Xi^3}\right)\right] \leftarrow = 0\\
&= \frac{\hat v_H^2}{4}\left[\frac{\delta(1-b_\Omega)b_\Omega}{\mu_\Omega^3} - \frac{\delta(1-\hat b_\Omega)\hat b_\Omega}{\hat\mu_\Omega^3} + \delta_{\Omega\hat\Omega}\sqrt{b_\Omega\hat b_\Omega}\left(\frac{1}{\mu_\Omega^3}+\frac{1}{\hat\mu_\Omega^3}\right)\right]\\
\mu_{sd} &= \frac{\hat v_H^2}{4}\left[\frac{\delta(1-b_\Omega)b_\Omega}{\mu_\Omega^3} - \frac{\delta(1-\hat b_\Omega)\hat b_\Omega}{\hat\mu_\Omega^3} + \delta_{\Omega\hat\Omega}\sqrt{b_\Omega\hat b_\Omega}\left(\frac{1}{\mu_\Omega^3}+\frac{1}{\hat\mu_\Omega^3}\right)\right.\\
&\qquad \left. -\frac{\delta(1-b_\Xi)b_\Xi}{\mu_\Xi^3} + \frac{\delta(1-\hat b_\Xi)\hat b_\Xi}{\hat\mu_\Xi^3} - \delta_{\Xi\hat\Xi}\sqrt{b_\Xi\hat b_\Xi}\left(\frac{1}{\mu_\Xi^3}+\frac{1}{\hat\mu_\Xi^3}\right)\right] \leftarrow = 0\\
&= \frac{\hat v_H^2}{4}\left[\frac{\delta(1-b_\Omega)b_\Omega}{\mu_\Omega^3} - \frac{\delta(1-\hat b_\Omega)\hat b_\Omega}{\hat\mu_\Omega^3} + \delta_{\Omega\hat\Omega}\sqrt{b_\Omega\hat b_\Omega}\left(\frac{1}{\mu_\Omega^3}+\frac{1}{\hat\mu_\Omega^3}\right)\right].
\end{aligned}
\tag{5.45}
$$

To write eqs. (5.43), (5.44) and (5.45), all bosonic and fermionic VEV's have been supposed to be real. This is in particular legitimate for all diagonal $< \bar q_i q_i >$ fermionic condensates of hermitian operators. We have no reasons a priori to consider that bosonic VEV's could become complex, and no sign either that $< \bar u c > = < \bar c u >$ and $< \bar d s > = < \bar s d >$ could get some imaginary parts.

Some explanations are due concerning the vanishing of the lines marked with arrows in (5.44), (5.45) above, which ensure in particular the symmetry relations $\mu_{uc} = \mu_{cu}, \mu_{ds} = \mu_{sd}$.
Among the terms with arrows stand for example $\frac{\delta(1-b_\Xi)b_\Xi}{\mu_\Xi^3}$. By (5.13), if we call the constant ratio $(1-b_X)/\nu_X^4 = \beta = (1-b_\Xi)/\nu_\Xi^4$, this term rewrites $\delta\frac{b_\Xi}{\mu_\Xi^3}\beta\nu_\Xi^4 = \delta\frac{b_\Xi}{\mu_\Xi^3}\beta\frac{2\mu_\Xi^6}{\hat v_\Xi^2} = \frac{2\delta b_\Xi\beta\mu_\Xi^3}{b_\Xi \hat v_H^2} = \frac{2\delta\beta\mu_\Xi^3}{\hat v_H^2}$; since $\mu_\Xi^3 = 0$, this term vanishes. So, $\frac{\delta b_\Xi(1-b_\Xi)}{\mu_\Xi^3} \stackrel{\mu_\Xi^3=0}{=} 0$ and, likewise, $\frac{\delta \hat b_\Xi(1-\hat b_\Xi)}{\hat\mu_\Xi^3} \stackrel{\hat\mu_\Xi^3=0}{=} 0$. Next, consider $\delta_{\Xi\hat\Xi}\sqrt{b_\Xi\hat b_\Xi}\left(\frac{1}{\mu_\Xi^3}\pm\frac{1}{\hat\mu_\Xi^3}\right)$. By the same argumentation one gets $\delta_{\Xi\hat\Xi}\sqrt{b_\Xi\hat b_\Xi}\left(\frac{1}{\mu_\Xi^3}\pm\frac{1}{\hat\mu_\Xi^3}\right) = \frac{\sqrt2\delta_{\Xi\hat\Xi}}{\hat v_H\sqrt\beta}\left(\sqrt{(1-b_\Xi)\hat b_\Xi} \pm \sqrt{(1-\hat b_\Xi)b_\Xi}\right)$, which vanish for $b_\Xi = 0 = \hat b_\Xi$. These properties ensure in particular that, as soon as their diagonal elements are real, the mass matrices of (u,c) and (d,s) quarks are hermitian [3].

5.6.2 Tuning θ_u to zero

Since $\tan 2\theta_u = -\frac{2\mu_{uc}}{\mu_u-\mu_c}$, tuning θ_u to 0 goes along with constraining μ_{uc} to vanish. Using the dimensionless b and r variables defined in (2.32) and (4.8), this is equivalent, by (5.44), to

$$
\theta_u = 0 \quad\Leftrightarrow\quad \delta_{\Omega\hat\Omega} = -\frac{\delta}{\sqrt{b_\Omega\hat b_\Omega}}\frac{1}{\hat r_\Omega - r_\Omega}\left(\hat r_\Omega b_\Omega(1-b_\Omega) + r_\Omega\hat b_\Omega(1-\hat b_\Omega)\right), \tag{5.46}
$$

which determines $\delta_{\Omega\hat\Omega}$ as a function of δ.

Tuning θ_u to 0 also entails that the mass eigenvalues are $\mu_u = m_u$ and $\mu_c = m_c$. By (5.43), this is equivalent to

$$
\delta_{X\hat X} = \frac{\delta}{\sqrt{b_X\hat b_X}(\hat r_X - \underbrace{r_X}_{1})}\left(\frac{2\sqrt2}{\rho}\underbrace{r_X}_{1}\hat r_X\frac{m_u}{m_u+m_d} - \hat r_X b_X(1-b_X) - \underbrace{r_X}_{1}\hat b_X(1-\hat b_X)\right), \tag{5.47}
$$

and

$$
\delta_{H\hat H} = \frac{\delta}{\sqrt{b_H\underbrace{\hat b_H}_{1}(\hat r_H - r_H)}}\left(\frac{2\sqrt2}{\rho}r_H\hat r_H\frac{m_c}{m_u+m_d} - \hat r_H b_H(1-b_H) - r_H\hat b_H\underbrace{(1-\hat b_H)}_{0}\right), \tag{5.48}
$$

[3] Then (see for example [12]), no bi-unitary transformation is needed to diagonalize them such that the quarks masses become identical to the eigenvalues of the mass matrices.

in which m_u, m_d, m_c are now considered as physical inputs and in which we have introduced the dimensionless variable ρ

$$\rho = \frac{\delta \hat{v}_H^2}{(m_u + m_d)\mu_X^3} \stackrel{(4.11)}{=} \frac{4\delta m_W^2}{g^2(m_u + m_d)\mu_X^3}\, \frac{1}{b_X + b_H + b_\Omega + 1 + \hat{b}_X + \hat{b}_\Omega}. \tag{5.49}$$

ρ is a function of $\delta, \hat{r}_H$ (and μ_X^3 but for this we can take its approximation (5.39)). Eqs. (5.47) and (5.48) therefore determine $\delta_{X\hat{X}}$ and $\delta_{H\hat{H}}$ as functions of δ and $\hat{r}_H$.

Taking (5.46), (5.47) and (5.48) into account, one gets now from (5.43) and (5.45)

$$\mu_d = -\frac{(m_u + m_d)\rho}{\sqrt{2}} \frac{b_X(1 - b_X) + \hat{b}_X(1 - \hat{b}_X)}{\hat{r}_X - 1} + m_u \frac{\hat{r}_X + 1}{\hat{r}_X - 1}, \tag{5.50}$$

$$\mu_s = -\frac{(m_u + m_d)\rho}{\sqrt{2}} \frac{b_H(1 - b_H)}{\hat{r}_H - r_H} + m_c \frac{\hat{r}_H + r_H}{\hat{r}_H - r_H}, \tag{5.51}$$

$$\mu_{ds} = -\frac{(m_u + m_d)\rho}{2} \frac{b_\Omega(1 - b_\Omega) + \hat{b}_\Omega(1 - \hat{b}_\Omega)}{\hat{r}_\Omega - r_\Omega}. \tag{5.52}$$

The masses of the d and s quarks are given by

$$\begin{aligned} m_d &= \frac{1}{2}\left((\mu_d + \mu_s) - \sqrt{(\mu_d - \mu_s)^2 + 4\mu_{ds}^2}\right), \\ m_s &= \frac{1}{2}\left((\mu_d + \mu_s) + \sqrt{(\mu_d - \mu_s)^2 + 4\mu_{ds}^2}\right), \end{aligned} \tag{5.53}$$

and the $\theta_d = \theta_c$ mixing angle by

$$\tan 2\theta_d = \frac{-2\mu_{ds}}{\mu_d - \mu_s}. \tag{5.54}$$

m_d, m_s and θ_c are accordingly functions of δ and $\hat{r}_H$.

5.7 Constraints of reality

The Yukawa Lagrangian (2.23) being hermitian, fermionic mass terms, in particular μ_d, μ_s, μ_{ds} cannot be but real. For $\theta_u = 0$, which translates into (5.46, 5.47, 5.48), their expressions are given in (5.50, 5.51, 5.52). These formulæ assume that all bosonic $b, \hat{b}$ and fermionic $\mu^3, \hat{\mu}^3$ VEV's are real, which a priori does not pose any problem, except for $\hat{\mu}_X^3$, because the value of $\hat{b}_X$ determined from the spectrum of pseudoscalar mesons is larger than 1: then, K^0 can only be orthogonal to $\bar{K}^0$ and π^0 to $K^0 + \bar{K}^0$ if $\hat{\mu}_X^3$ becomes imaginary.

Therefore, we have, at least formally, to allow $\hat{\mu}_X^3$ to become complex, which has consequences on μ_u and μ_d in (5.43).

5.7.1 Reality of μ_d

Because $\mu_d + \mu_s = m_d + m_s$ and since μ_s, as we shall see in subsection 5.7.4), has no problem of reality, μ_d should stay real whatever happens to $\hat{\mu}_X^3$. We accordingly constrain its imaginary part to vanish.

All desired orthogonality conditions are supposed to be satisfied. In particular (5.20) is equivalent to $\hat{b}_X(1-\hat{b}_X) = 2\hat{r}_X^2 b_X(1 - b_X)$; therefore, $\hat{r}_X^2$ is real, and $\hat{r}_X^2 - 1$ too. Thanks to this, one easily gets from (5.50)

$$\Im(\mu_d) = \frac{1}{\hat{r}_X^2 - 1}\left[-\frac{(m_u + m_d)\rho}{\sqrt{2}} b_X(1 - b_X)(1 + 2\hat{r}_X^2)\Im\hat{r}_X + 2m_u \Im\hat{r}_X\right], \tag{5.55}$$

such that the reality condition of μ_d reads

$$\mu_d\; real\; \Leftrightarrow 2m_u = \frac{(m_u + m_d)\rho}{\sqrt{2}}\Big(b_X(1 - b_X) + \hat{b}_X(1 - \hat{b}_X)\Big) = \frac{\delta \hat{v}_H^2}{\sqrt{2}\mu_X^3}\Big(b_X(1 - b_X) + \hat{b}_X(1 - \hat{b}_X)\Big). \tag{5.56}$$

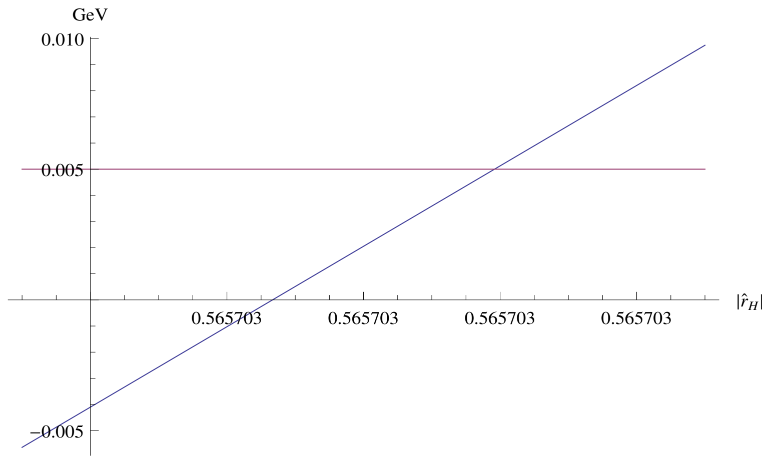

Figure 5.2: The r.h.s. of (5.56) is plotted as a function of $|\hat r_H|$ for $\delta \approx m^2_{D_s}$ and $\hat v_H \approx 143\,GeV$; the horizontal line is the value of the l.h.s. $2m_u \approx 5\,MeV$

Note that this entails in particular that $b_X(1-b_X)+\hat b_X(1-\hat b_X)$ is a very small positive number (we suppose that $m_u>0$). When (5.56) is realized, one also gets from (5.50)

$$\mu_d = \frac{1}{\hat r_X^2-1}\left(\underbrace{-\frac{m_u+m_d}{\sqrt 2}\rho\, b_X(1-b_X)(1+2\hat r_X^2)}_{-2m_u}(1+\Re\hat r_X)+m_u(1+\hat r_X^2+2\Re\hat r_X)\right) = m_u\frac{-1+\hat r_X^2}{\hat r_X^2-1}. \tag{5.57}$$

So,

$$\mu_d\; real\; \Rightarrow \mu_d = m_u, \tag{5.58}$$

which goes accordingly to 0 at the chiral limit. When (5.56) is realized, one also gets from (5.47)

$$\delta_{X\hat X} = \delta\frac{\hat b_X(1-\hat b_X)}{\sqrt{b_X\hat b_X}}. \tag{5.59}$$

5.7.2 First consequence: determination of $|\hat r_H|$

The determination of $\hat r_H$ is done through (5.56). We plot in Fig. 5.2 its r.h.s. in which we have inserted the estimate (5.42) for $\hat v_H$ and $\delta \approx m^2_{D_s}$, as a function of $|\hat r_H|$. The horizontal red line is the value of $2m_u \approx 5\,MeV$. In practice, due to the smallness of m_u and the large value of $\delta\hat v_H^2/\sqrt 2\mu_X^3$, the solution of this equation is practically the same as that of $b_X(1-b_X)+\hat b_X(1-\hat b_X)=0$. It is very precise, with furthermore an extremely small sensitivity to variations of δ and of $\hat v_H$.

One gets

$$|\hat r_H| \approx .565703. \tag{5.60}$$

A reasonable approximate value can be obtained from the expansions of b_X and $\hat b_X$ (5.30) and (5.29) at the chiral limit. They yield $b_X(1-b_X)+\hat b_X(1-\hat b_X)\approx 0$ at $\hat r_H^2 = 1/3 = (.58)^2$.

5.7.3 Second consequence: another determination of the mixing angle

Once (5.58) has been implemented, (5.53) and (5.54) yield

$$m_d = m_u\frac{1+\sqrt{1+\tan^2 2\theta_c}}{2} + \mu_s\frac{1-\sqrt{1+\tan^2 2\theta_c}}{2}. \tag{5.61}$$

If we believe that $m_d > 0$ and that, at least, $m_d \geq 1.5\, m_u$, we are having troubles with (5.61). One indeed needs the second term in its r.h.s. to be positive; but, the numerator of the fraction being negative, this requires $\mu_s < 0$. However, $m_d + m_s = \mu_d + \mu_s$ becomes, due to (5.56), $m_d + m_s = m_u + \mu_s$ such that

$$\mu_s = m_d + m_s - m_u \tag{5.62}$$

has to be positive. The only solution is therefore

$$m_d < 0 \tag{5.63}$$

like we have found for 1 generation. Using (5.61) and (5.62) leads straightforwardly to the result

$$\tan^2\theta_d = \frac{m_u - m_d}{m_s - m_u} = \frac{m_u + |m_d|}{m_s - m_u}, \tag{5.64}$$

which is, like (5.1) a fairly accurate formula if one takes [20] $m_u \approx 2.5\, MeV, |m_d| \approx 5\, MeV, m_s \approx 100\, MeV$. It is certainly not a new output since similar estimates have already been obtained (see for example [8] [9] [10]). However, the way it has been obtained is new.

Note that a consequence of (5.62) is that, at the chiral limit $m_u, m_d \to 0$, $\mu_s \to m_s$. Furthermore, since we know that $m_u, |m_d| \ll m_s$, at a good approximation $\mu_s \approx m_s$.

5.7.4 Reality of μ_s

Since $\hat r_H$ has been shown to be non-pathological (we will determine its sign later), the reality of μ_s as given by (5.51) can only be put in jeopardy if r_H becomes complex, which means, by (5.15), if $b_H < 0$. This would correspond to a physically unacceptable imaginary v_H (once $\hat v_H$ is indeed real). Despite this reservation, we may play the same game as for μ_d and require that, when b_H formally becomes negative, the imaginary part of μ_s should vanish.

$$\mu_s = \frac{1}{\hat r_H^2 - r_H^2}\left[-(\hat r_H + r_H) b_X(1-b_X) r_H^2 \frac{(m_u+m_d)\rho}{\sqrt{2}} + m_c(\hat r_H^2 + r_H^2 + 2 r_H \hat r_H)\right], \tag{5.65}$$

such that

$$\Im\mu_s = \frac{\Im r_H}{\hat r_H^2 - r_H^2}\left[-\frac{(m_u+m_d)\rho}{\sqrt{2}} b_H(1-b_H) + 2 m_c \hat r_H\right], \tag{5.66}$$

and

$$\Im\mu_s = 0 \Leftrightarrow 2m_c = \frac{(m_u+m_d)\rho}{\sqrt{2}}\frac{b_H(1-b_H)}{\hat r_H} = \frac{(m_u+m_d)\rho}{\sqrt{2}}\frac{r_H^2 b_X(1-b_X)}{\hat r_H} = \frac{\delta \hat v_H^2}{\sqrt{2}\mu_X^3}\frac{r_H^2 b_X(1-b_X)}{\hat r_H}. \tag{5.67}$$

When this is realized

$$\mu_s = \Re\mu_s = \frac{1}{\hat r_H^2 - r_H^2}\left[-(\hat r_H + \Re r_H)\underbrace{b_X(1-b_X) r_H^2 \frac{(m_u+m_d)\rho}{\sqrt{2}}}_{2m_c \hat r_H} + m_c(\hat r_H^2 + r_H^2 + 2\hat r_H \Re r_H)\right] = -m_c. \tag{5.68}$$

This result being in flagrant disagreement with what we have obtained before, in particular at the chiral limit at which $\mu_s = m_s$, we conclude that a negative b_H is totally excluded, which requires, as we have already written in (5.9), $\delta \geq 3.90923\, GeV^2$. It will be satisfied by our result in (6.14).

5.8 The fermionic mixing angle : a paradox

Using (5.56) and (5.62), (5.54) rewrites (we recall $\theta_d = \theta_c$)

$$\tan 2\theta_d = \frac{2\mu_{ds}}{m_d + m_s - 2m_u}, \tag{5.69}$$

or, equivalently

$$\mu_{ds} = (m_d + m_s - 2m_u)\frac{\tan\theta_c}{1-\tan^2\theta_c}. \tag{5.70}$$

On the other side, μ_{ds} is given by (5.52) and (5.49)

$$\mu_{ds} = \frac{\delta \hat{v}_H^2}{2\mu_X^3}\, \frac{b_\Omega(1-b_\Omega) + \hat{b}_\Omega(1-\hat{b}_\Omega)}{r_\Omega - \hat{r}_\Omega}. \tag{5.71}$$

Neither in (5.70) nor in (5.71) did we make use of any bosonic relation, except those in (2.31) of the type $\delta_i = \delta(1-b_i)$ which result from the minimization of the effective potential. At this point (5.15), relating the r's to the b's, has not been used, nor any of the bosonic relations connecting the b's to δ and to the masses of pseudoscalar mesons. In this respect the mixing angle occurring in (5.71) appears as the "fermionic mixing angle".

$\delta\hat{v}_H^2/2\mu_X^3$ which occurs in the r.h.s. of (5.71) is a very large scale while, a priori, at least from bosonic considerations and measurements, we expect a small (bosonic) θ_c, which, by (5.70), requires a small μ_{ds}.

Then, to reconcile (5.70) and (5.71) one needs:
* either a very small value for $\frac{b_\Omega(1-b_\Omega)+\hat{b}_\Omega(1-\hat{b}_\Omega)}{r_\Omega - \hat{r}_\Omega}$;
* or a "fermionic mixing angle" which is different from the bosonic mixing angle and which is close to maximal ($\pi/4$) to enlarge the r.h.s. of (5.70).

Even if the b's are probably subject to uncertainties, it is very unlikely that b_Ω and $\hat{b}_\Omega$ are in reality close to 0 or 1. If they stay $\mathcal{O}(1)$ as we determined from bosonic considerations, then the only solution concerning (5.71) is that $r_\Omega - \hat{r}_\Omega \equiv \sqrt{2}\frac{<\bar{d}s+\bar{s}d>}{<\bar{u}u+\bar{d}d>}$ becomes very large.

Let us now consider the bosonic evaluation of this quantity. By (5.14), $r_\Omega - \hat{r}_\Omega = \frac{\sqrt{b_\Omega(1-b_\Omega)} - \sqrt{\hat{b}_\Omega(1-\hat{b}_\Omega)}}{\sqrt{b_X(1-b_X)}}$ and we have found that b_Ω is not very different from $\hat{b}_\Omega$ [4]. It is thus very unlikely that $r_\Omega - \hat{r}_\Omega$ becomes very large unless b_X comes extremely close to 1, much closer than what we found from bosonic considerations.

In this case, sticking to a small mixing angle, the r.h.s. of (5.71) can only match the one of (5.70) if very little confidence can be attached to the bosonic determinations of the b's, in particular that of b_X. One knows that $b_X \to 1$ at the chiral limit $(m_u, m_d, m_\pi \to 0, \mu_X^3\ fixed)$, and we may wonder whether our calculations could be, for some unknown reason, only valid at this limit. This is far from satisfying, because then, since the Cabibbo angle also vanishes at this limit (see (5.1)), no credit whatsoever should be then granted to our calculations (5.54) and (5.64). This is why we look for other possibilities:
* it may happen that adding the 3rd generation of quarks yield a value of b_X naturally much closer to 1;
* the r.h.s. of (5.71) is instead doomed to stay naturally large, such that its matching with the r.h.s. of (5.70) calls for a quasi-maximal fermionic mixing angle $\tan^2\theta_d \approx 1$. This reminds of leptons, where large fermionic mixing angle(s) seem to be "natural". There, they are directly measured from the corresponding asymptotic states, which is not the case for mesons.

At this stage, the situation for 2 generations can only be summarized as follows: *the "bosonic" evaluation of μ_{ds} (we mean by this the evaluation of the r.h.s. of (5.71) in which the b's and r's are calculated from bosonic considerations) corresponds to a quasi-pole of its fermionic expression (5.70) and, accordingly, to $\theta_d^{fermionic} \approx \pi/4$.*

[4] All other masses and parameters being left untouched, moving the mass of the neutral D^0 meson from $m_{D^0} = 1864.86\,MeV$ [20] down to $m_{D^0} = 1862.29\,MeV$ is enough to match $b_\Omega = \hat{b}_\Omega$, that is, to reach the pole of μ_{ds}.

We shall see in chapter 6 that, in reality, the situation already largely improves when one switches on $\theta_u \neq 0$, which in particular brings b_Ω (and presumably also $\hat{b}_\Omega$) very close to 0. But this cannot be guessed at this stage of the study.

5.9 The chiral limit and the s quark mass; the sign of $\hat{r}_H$

By going to the chiral limit $m_u, m_d, m_\pi \to 0$, with $< \bar{u}_m u_m + \bar{d}_m d_m >$ fixed, we shall show that $\hat{r}_H \equiv \frac{<\bar{c}c - \bar{s}s>}{<\bar{u}u + \bar{d}d>}$ is positive and get an estimate of how close δ stays to $m^2_{D_s}$.

We know by (5.1) that the Cabibbo angle goes to 0 at the chiral limit $m_\pi \to 0$. This also means by (5.54) that, at the fermionic level, $\mu_{ds} \to 0$, such that in particular, $\mu_s \simeq m_s$ as given by (5.51). According to (5.49), $(m_u + m_d)\rho \equiv \frac{4\delta m_W^2}{g^2 \mu_X^3} \frac{1}{b_X + b_H + b_\Omega + 1 + \hat{b}_X + \hat{b}_\Omega}$ is a very large mass scale. Indeed, we already know that $\delta \geq m^2_{D_s}$, $g = \mathcal{O}(1)$, μ_X^3 is supposed to stay constant at the chiral limit and keep a value close to its physical value as given by the GMOR relation, and the b's are of order 1 or smaller. So, the role of $(m_u + m_d)\rho$ in (5.51) should be damped by b_H becoming very small, which, as we saw on (5.8), can only happen for $\delta \to m^2_{D_s}$.

Let us be more precise. When $m_\pi \to 0$,

$$\begin{aligned} b_X(1-b_X) &\simeq \frac{m_\pi^2}{\delta} + \frac{m_\pi^4}{\sigma^4} + \ldots, \\ with \quad \frac{1}{\sigma^4} &= -\frac{1}{\delta}\left(\frac{1}{\delta} - \frac{1}{m^2_{D^+}} + \frac{1}{m^2_{K^+}}\right); \end{aligned} \tag{5.72}$$

$$\begin{aligned} b_H(1-b_H) &\simeq \frac{m^2_{D_s}(\delta - m^2_{D_s})}{\delta^2} - \frac{m_\pi^2}{\tau^2} + \ldots, \\ with \quad \tau^2 &= \frac{\delta^2 m_D^4 m_K^2}{m^2_{D_s}(\delta - 2m^2_{D_s})(m_D^2 - m^2_{D_s})(m_D^2 - m_K^2)} \stackrel{\delta \approx m^2_{D_s}}{\approx} \frac{m_D^4 m_K^2}{(m^2_{D_s} - m_D^2)(m_D^2 - m_K^2)} + \ldots. \end{aligned} \tag{5.73}$$

(5.15), (5.72) and (5.73) yield

$$r_H^2 = \frac{b_H(1-b_H)}{b_X(1-b_X)} \approx \frac{\frac{m^2_{D_s}(\delta - m^2_{D_s})}{\delta^2} - \frac{m_\pi^2}{\tau^2}}{\frac{m_\pi^2}{\delta} + \underbrace{\frac{m_\pi^4}{\sigma^4}}_{can\ be\ neglected}} \approx \frac{m^2_{D_s}(\delta - m^2_{D_s}) - \delta^2 \frac{m_\pi^2}{\tau^2}}{\delta m_\pi^2} \approx \frac{m^2_{D_s}(\delta - m^2_{D_s})}{\delta m_\pi^2} - \frac{\delta}{\tau^2}. \tag{5.74}$$

Then, from (5.51) one gets for δ close to $m^2_{D_s}$

$$m_s \simeq \mu_s \approx \frac{-\frac{(m_u+m_d)\rho}{\sqrt{2}}\left(\frac{m^2_{D_s}(\delta - m^2_{D_s})}{\delta^2} - \frac{m_\pi^2}{\tau^2}\right) + m_c\left(\hat{r}_H + \sqrt{\frac{m^2_{D_s}(\delta - m^2_{D_s})}{\delta m_\pi^2} - \frac{\delta}{\tau^2}}\right)}{\hat{r}_H - \sqrt{\frac{m^2_{D_s}(\delta - m^2_{D_s})}{\delta m_\pi^2} - \frac{\delta}{\tau^2}}}. \tag{5.75}$$

Inside the $\sqrt{\ }$'s an in-determination arises when both $m_\pi \to 0$ and $\delta \to m^2_{D_s}$. To lift it, the following limit has to be taken

$$\delta \stackrel{m_\pi \to 0}{\sim} m^2_{D_s} + \omega^2 m_\pi^2. \tag{5.76}$$

Plugging (5.76) into (5.75) yields

$$m_s \approx \mu_s \stackrel{m_\pi \to 0}{\approx} \frac{-\frac{(m_u+m_d)\rho}{\sqrt{2}}\left(\frac{\omega}{m^2_{D_s}} - \frac{1}{\tau^2}\right) m_\pi^2 + m_c\left(\hat{r}_H + \sqrt{\omega - \frac{\delta}{\tau^2}}\right)}{\hat{r}_H - \sqrt{\omega - \frac{\delta}{\tau^2}}}. \tag{5.77}$$

We display in Fig.5.3 the curve giving m_s as a function of ω for $\hat{r}_H = +.59215 > 0$, which is the only sign that can fit the value of m_s. We truncated the range of ω so as to avoid the pole that appears in (5.77) in this case. The c quark mass has been taken as $m_c = 1.2755\, GeV$ [20].

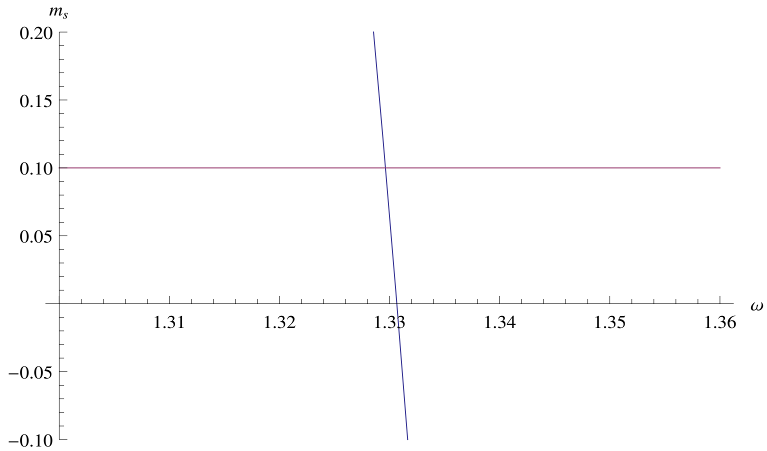

Figure 5.3: The chiral limit of m_s as a function of ω for $\hat{r}_H = +59215$ and $m_c = 1.2755\,GeV$

This determines $\omega \approx 1.33$, together with

$$\hat{r}_H \equiv \frac{< \bar{c}c - \bar{s}s >}{< \bar{u}u + \bar{d}d >} \approx +.565703, \tag{5.78}$$

and, by (5.76),

$$\delta = m^2_{D_s} + (1.33\, m_\pi)^2 \approx 3.9094\, GeV^2. \tag{5.79}$$

The value (5.79) is very close to the lower bound (5.9) below which b_H becomes negative.

Strictly speaking, we have not proved that, for physical pions, δ stays close to this value and does not become much larger. However, the parameters of all formal expansions that we have been using $(\frac{m_\pi^2}{m_K^2}, \frac{m_\pi^2}{m_D^2}, \frac{m_\pi^2}{m_{D_s}^2}, \frac{m_\pi^2}{\delta})$ are presumably small enough for a good convergence and we think reasonable, at this stage, not to expect large deviations from the value (5.79).

5.10 Summarizing the solutions of the equations

The set of solutions includes (5.78) and (5.79). We also recall (5.1)

$$t^2 \equiv \tan^2\theta_d = \frac{\dfrac{1}{m^2_{K^\pm}} - \dfrac{1}{m^2_{D^\pm}}}{\dfrac{1}{m^2_{\pi^\pm}} - \dfrac{1}{m^2_{D_s^\pm}}} \approx .07473 \Rightarrow \theta_c \approx .2668. \tag{5.80}$$

Plugging (5.78) and (5.79) in (5.5), (5.15), (5.24), (5.27), (5.41) yield

$$\begin{aligned} &b_X \approx .99536, \quad b_H \approx 4.64\;10^{-5}, \quad b_\Omega \approx .20289, \quad b_\Xi = 0,\\ &\hat{b}_X \approx 1.0046, \quad \hat{b}_H = 1, \quad \hat{b}_\Omega \approx .19486, \quad \hat{b}_\Xi = 0. \end{aligned} \tag{5.81}$$

$$\hat{v}_H \approx 142.973\; GeV. \tag{5.82}$$

Together with (5.81), this yields

$$v_X \approx 142.641\,GeV, \quad \hat{v}_X \approx 143.301\,GeV, \quad v_H \approx 973\,MeV, \quad v_\Omega \approx 64.40\,GeV, \quad \hat{v}_\Omega \approx 63.11\,GeV. \tag{5.83}$$

One has also found

$$\begin{aligned} &r_X = 1, \quad r_H \approx .1002, \quad r_\Omega \approx 5.9187, \quad r_\Xi = 0,\\ &\hat{r}_X =?, \quad \hat{r}_H \approx .59215, \quad \hat{r}_\Omega \approx 5.8295, \quad \hat{r}_\Xi = 0. \end{aligned} \tag{5.84}$$

The following relations have been obtained from fermionic considerations. m_d has been determined to be negative, in particular from the expression of the mixing angle (5.64)

$$\tan^2\theta_d = \frac{m_u - m_d}{m_s - m_u} = \frac{m_u + |m_d|}{m_s - m_u} \approx .076923 \Rightarrow \theta_c \approx .2705, \tag{5.85}$$

in good agreement with the value extracted from (5.54) (both leading however to a slightly too large value for the Cabibbo angle (1.6)). The quark masses $m_u \approx 2.5\, MeV, m_d \approx -5\, MeV, m_s \approx 100\, MeV, m_c \approx 1.2755\, GeV$ have been used as inputs. In particular, from the values of m_u, m_d, the GMOR relation yielded the value of μ_X^3

$$m_u \approx 2.5\ MeV,\ m_d \approx -5\ MeV \quad \Rightarrow \quad \frac{<\bar u u + \bar d d>}{\sqrt{2}} = \mu_X^3 \stackrel{(5.39)}{\approx} \frac{\sqrt{2} f_\pi^2 m_{\pi^+}^2}{m_u + m_d} \approx -0.09370\ GeV^3. \tag{5.86}$$

From the values (5.81) of the b's we deduce by (2.38) and (2.39) the masses of the Higgs bosons

$$\begin{gathered} m_{X^0} \approx 2.7897\, GeV \qquad m_{\hat H^3} \approx 2.7962\, GeV \qquad m_{\hat X^3} \approx 2.8026\, GeV \\ m_{\Omega^0} \approx 1.2595\, GeV \qquad m_{\hat\Omega^3} \approx 1.2343\, GeV \\ m_{H^0} \approx 19\, MeV \\ m_{\Xi^0}\ small \qquad m_{\hat\Xi^3}\ small \end{gathered} \tag{5.87}$$

From the values (5.84) of the r's and their definition (4.8) we deduce the fermionic VEV's

$$\begin{aligned} <\bar c c> &= \frac{r_H + \hat r_H}{\sqrt{2}}\, \mu_X^3 \approx .47\, \mu_X^3 < 0, \\ <\bar s s> &= \frac{r_H - \hat r_H}{\sqrt{2}}\, \mu_X^3 \approx -.33\, \mu_X^3 > 0, \\ <\bar u c> = <\bar c u> &= \frac{r_\Omega + \hat r_\Omega}{2}\, \mu_X^3 \approx 5.87\, \mu_X^3 < 0, \\ <\bar d s> = <\bar s d> &= \frac{r_\Omega - \hat r_\Omega}{2}\, \mu_X^3 \approx .045\, \mu_X^3 < 0. \end{aligned} \tag{5.88}$$

We notice a large non-diagonal $<\bar u c> = <\bar c u>$ condensation, which could certainly not be predicted on perturbative grounds since it starts occurring only at 2-loops. This result will get modified when θ_u is no longer approximated by 0.

As far as $\hat\mu_X^3 = \frac{<\bar u u - \bar d d>}{\sqrt{2}}$ is concerned, we cannot find any acceptable (real) value compatible with the orthogonality of K^0 to $\bar K^0$, and of π^0 to $K^0 + \bar K^0$ since these 2 conditions are associated with (5.19) while one has found $\hat b_X > 1$ from the ratio $m_{K^0}^2 / m_{D^0}^2$.

5.10.1 The small value of b_H versus $b_X \approx \hat b_X \approx \hat b_H = 1$

It would be desirable to have an analytic expression for b_H which reflects its small value. It unfortunately turns out that its complete analytical expression in terms of δ and charged pseudoscalar masses, despite its relative simplicity, has no trustable expansion in powers of m_π at the chiral limit $m_\pi \to 0$. The most meaningful expansion that we could get is by writing $\delta = \delta_{(b_H=0)} + \epsilon\, m_\pi^2$, in which $\delta_{(b_H=0)}$ is the value of δ at which b_H vanishes, given by the r.h.s. of the first line of (5.8) and $\epsilon = \frac{\delta - \delta_{(b_H=0)}}{m_\pi^2} \stackrel{(5.76,5.79)}{=} \omega^2 + \frac{m_{D_s}^2 - \delta_{(b_H=0)}}{m_\pi^2} \approx .0093$. One gets then

$$b_H \equiv \frac{v_H}{\hat v_H} \approx \epsilon \frac{m_\pi^2}{m_{D_s}^2}, \tag{5.89}$$

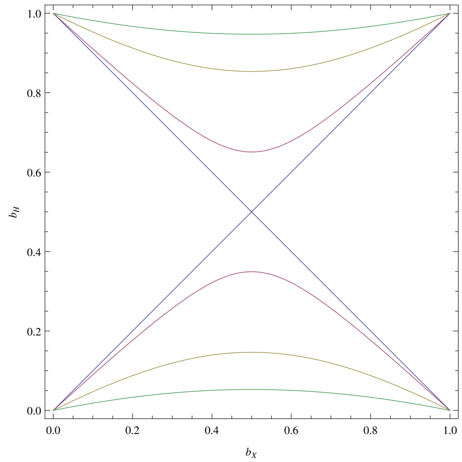

Figure 5.4: b_H at $\theta_u = 0$ as a function of b_X for $1/r_H^2 = 1$ (blue), 1.1 (purple), 2 (yellow) and 5 (green)

which exhibits a sort of "see-saw" mechanism between the two scales m_π and m_{D_s}, strengthened by the small factor ϵ. Though $\omega = \mathcal{O}(1)$, ϵ gets $\ll 1$ because of two near cancellations: the first occurs between $m_{D_s}^2$ and $\delta_{(b_H=0)}$, and the second between ω^2 and $\frac{m_{D_s}^2 - \delta_{(b_H=0)}}{m_\pi^2}$. Eq. (5.89) yields $b_H \to 0$ at the chiral limit, which is compatible with the exact result $b_H \stackrel{m_\pi \to 0}{\longrightarrow} 1 - \frac{m_{D_s}^2}{\delta}$ since, by (5.76), $\delta \stackrel{m_\pi \to 0}{\longrightarrow} m_{D_s}^2$.

A "see-saw" mechanism between b_H and b_X can be traced back to the orthogonality relations between charged pseudoscalars (4.17), which entail in particular (see (5.15)).

$$b_X(1-b_X) = \frac{b_H(1-b_H)}{r_H^2}, \quad r_H = \frac{< \bar c c + \bar s s >}{< \bar u u + \bar d d >}. \tag{5.90}$$

Since $b_X \approx 1 \Rightarrow v_X \approx \hat v_H$, the same phenomenon occurs between b_H and $\hat b_H$. It is due to the property $|r_H| < 1$ which one can understand by the fact that heavy quarks being more "classical" that light ones, they are expected to condense less. One can indeed easily check that

$$|r_H| \ll 1 \Rightarrow b_H \ll b_X \; or \; b_H \gg b_X. \tag{5.91}$$

It is easily visualized on Fig.5.4 in which we plot b_H as a function of b_H for $1/r_H^2 = 1$ (blue), $1/r_H^2 = 1.1$ (purple), $1/r_H^2 = 2$ (yellow) and $1/r_H^2 = 5$ (green). Since we found $1/r_H^2 \approx 100$ (see (5.84)), the see-saw mechanism is very effective.

5.10.2 Hierarchies

The largest hierarchy among bosonic VEV's is $1/\sqrt{b_H} \approx 151$. It is already much smaller that the one occurring for 1 generation, that we recall (see (3.17)) to be ≈ 2858. The other ones, which are given by $1/\sqrt{b}$'s are all of $\mathcal{O}(1)$.

Fermionic hierarchies are given by the r's and do not exceed $r_\Omega \approx 5-6$. Since it is hard in the 2-generation case to have a reliable calculation of $< \bar d d > / < \bar u u >$, one cannot draw, yet, definitive conclusions.

Seemingly, when more generations are added, the general trends is a decrease of the hierarchies among VEV's. This can be easily understood because large masses, like that of gauge bosons, get "shared" by several VEV's, and so are the masses of heavy quarks. The number of Higgs bosons growing like $2N^2$, one can naively expect that, for example, bosonic hierarchies are 8 times smaller for 2 generations than for 1 generation. This is of course just an estimate, and we have seen that the decrease is even stronger.

If this trends goes on, we can expect still smaller hierarchies for 3 generations, because large mass scales will be shared among 18 quadruplets. The "$\tan\beta$" which was disquietingly huge for 1 generation could then become replaced by a series of much smaller and more natural numbers.

5.10.3 The light Higgs bosons H^0, Ξ^0 and $\hat\Xi^3$

Like for 1 generation, a light Higgs H^0 arises in the Higgs quadruplet that is parity transformed of the one that contains the 3 Goldstones of the broken $SU(2)_L$. Its mass lies well below that of the lightest pseudoscalars such that, in particular, it cannot decay into 2 pions.

As far as Ξ^0 and $\hat\Xi^3$ are concerned, we could not calculate explicitly their masses, but there are reasonable argument that they should not vanish nor become large. They should not vanish because they do not correspond to any true Goldstone boson. Also, because the hypothesis $<\bar c u>=<\bar u c>$ which led to their classical vanishing for 2 generations should no longer be true with 3 generations. To have an intuitive idea of this, at least at the perturbative level, it is enough to realize that such condensates occur at 2-loops with an intermediate gauge boson line, and will accordingly be sensitive to the CP violating phase(s) that cannot be avoided in this case. Their effects are expected to be small.

Next, even for 2 generations, one expects quantum corrections to their classical masses through fermion loops as follows. Let us note $(\bar q_i q_j)^*$ the fermion loop with quarks $\bar q_i$ and q_j. From the Yukawa Lagrangian, one gets, at second order, the couplings $\delta^2_{\Xi\hat\Xi}\Big(\Xi^0\frac{\hat v_\Xi}{\sqrt2\hat\mu^3_\Xi}\frac14(\bar uc-\bar cu-\bar ds+\bar sd)^*\frac{\hat v_\Xi}{\sqrt2\hat\mu^3_\Xi}\Xi^0+\hat\Xi^3\frac{v_\Xi}{\sqrt2\mu^3_\Xi}\frac14(\bar uc-\bar cu+\bar ds-\bar sd)^*\frac{v_\Xi}{\sqrt2\mu^3_\Xi}\hat\Xi^3\Big)$. In section 5.2, we have emphasized that, at $b_\Xi=0$, the normalization $v_\Xi/\sqrt{2\mu^3_\Xi}$ of the Ξ quadruplet is a constant, and so is the one of the $\hat\Xi$ quadruplet. They are the same as for the X quadruplet, $1/\nu^2_\Xi=1/\hat\nu^2_\Xi=1/\nu^2_X=\hat v_H\sqrt{b_X(1-b_X)}/\sqrt2\mu^3_X$. One gets accordingly the couplings $\frac{\delta^2_{\Xi\hat\Xi}}{4\nu^4_X}\Big(\Xi^0(\bar uc-\bar cu-\bar ds+\bar sd)^*\Xi^0+\hat\Xi^3(\bar uc-\bar cu+\bar ds-\bar sd)^*\hat\Xi^3\Big)$. The same (uc) and (ds) fermion loops occur for Ξ^0 and $\hat\Xi^3$; they are accordingly expected to yield the same mass terms for Ξ^0 and $\hat\Xi^3$. This is of course only an intuitive perturbative argument. In particular the fermion loops are quadratically divergent and need to be regularized. One can also notice that the argument is in some way iterative because the fermion loops can themselves be "saturated" by virtual $\hat\Xi^3$ and Ξ^0 lines.

5.11 The masses of π^0, K^0 and D^0; tracing why $\hat b_X=(\hat v_X/\hat v_H)^2>1$ is needed

The physical value $\hat b_X>1$ that we determined from the spectrum of pseudoscalar mesons is problematic because it is in contradiction with the 2 identical relations (5.20) and (5.18) that control the orthogonality of π^0 to $K^0+\bar K^0$ and the absence of non-diagonal $K^0-\bar K^0$ terms. We now look deeper for the origin of this problem.

In addition to (4.24) and (4.25) for the masses of neutral kaon and D mesons, one gets for the neutral pion (we recall that its interpolating field we chose proportional to $\bar u\gamma_5 u-\bar d\gamma_5 d$)

$$m^2_{\pi^0}=\frac{2\delta(1+s_d^2)}{\dfrac{(1+c_d^2)^2}{1-b_X}+2\dfrac{(1-c_d^2)^2}{1-\hat b_X}+\dfrac{s_d^4}{1-b_H}+\dfrac{s_d^4}{b_X(1-b_X)}\dfrac{1}{\hat r^2_H}+2s_d^2c_d^2\left(\dfrac{1}{1-b_\Omega}+\dfrac{1}{1-\hat b_\Omega}\right)}. \tag{5.92}$$

At the values of the parameters that we determined (see section 5.10), one gets

$$m_{D^0}\approx 1.865\,GeV,\quad m_{K^0}\approx 497.61\,MeV,\quad m_{\pi^0}\approx 139.38\,MeV, \tag{5.93}$$

which is quite satisfying and shows in particular that these values of the parameters can not only nicely fit $m^2_{K^0}/m^2_{D^0}$ as we did in subsection 5.3.2, but also the absolute masses of the neutral mesons.

Then the question arises: why are we induced to the troublesome $\hat b_X>1$? The numerical analysis shows that, while the value of m_{D^0} is quite stable for $\hat b_X$ close to 1, it is not the case for m_{K^0}. For $\hat b_X<1$ one gets too small a value of m_{K^0}, which, however, turns out to have a pole at $\hat b_X$ slightly above 1 (see Fig.5.5). The suitable mass can only be recovered above the pole, which is of course a very artificial and ad-hoc solution.

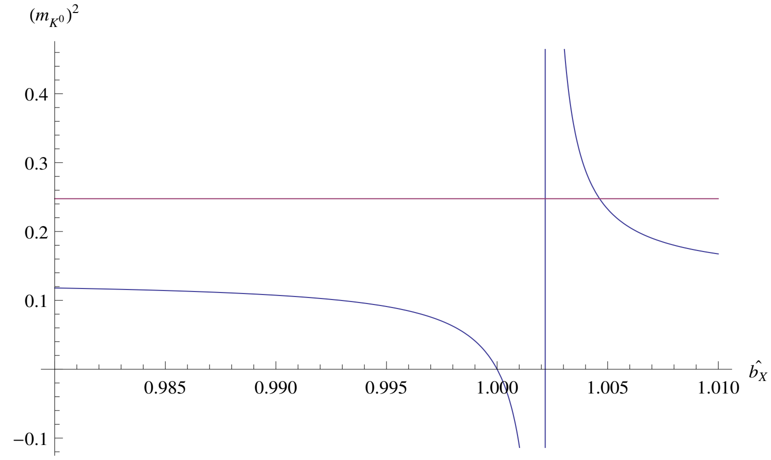

Figure 5.5: $m^2_{K^0}$ as a function of $\hat{b}_X$ at $\theta_u = 0$. The other parameters are fixed to their determined values. The horizontal line is at the physical value of $m^2_{K^0}$

To give an idea of by how much m_{K^0} is found too small, we give the values of the same masses at $\hat{b}_X = .95$: $m_{D^0} \approx 1.865\, GeV, \quad m_{K^0} \approx 354\, MeV, \quad m_{\pi^0} \approx 139.18\, MeV$.

The conclusion is therefore that the masses of charged mesons $\pi^\pm, K^\pm, D^\pm, D_s^\pm$, and those of π^0 and D^0 can be accounted for very "naturally", but, then, the mass of the K^0 meson is short by $140\, MeV$ (nearly the pion mass), unless one goes to $\hat{b}_X > 1$. This unnatural solution raises then another issue connected to the orthogonality of π^0 to $K^0 + \bar{K}^0$. Indeed, (5.18) yields then a complex unrealistic value for $\hat{r}_X \equiv < \bar{u}u - \bar{d}d > / < \bar{u}u + \bar{d}d >$.

5.11.1 Can one restore $\hat{b}_X < 1$?

One may question the way we have defined the π^0 interpolating field and/or the necessity to cancel $K^0\bar{K}^0$ non-diagonal mass terms (this is akin to giving up the neutral kaon mass eigenstates as $K^0 \pm \bar{K}^0$). The mixing among neutral mesons may also be more subtle than usually thought of, and the whole set of interpolating fields that we have chosen be much too naive.

There is most probably no need to go to such extremes because, as we shall see in chapter 6, the situation largely improves when one allows for $\theta_u \neq 0$. This goes however with a value $\hat{r}_H \geq .945$ larger than the one that we found here, which means that $< \bar{c}c - \bar{s}s > \simeq < \bar{u}u - \bar{d}d >$. So, large condensates of heavy quarks are required. A hint in this direction appears in the expansions (5.29) and (5.30). Since (5.30) depends on the sole parameter δ, $b_X \leq 1$ looks a robust property. This is however not quite the case for $\hat{b}_X$ because its expansion (5.29) also depends on the ratio $\hat{r}_H$ of fermionic VEV's and as soon as $|\hat{r}_H| \geq 1$ (see also footnote 2) $\hat{b}_X$ becomes ≤ 1 as desired. $|\hat{r}_H| \geq 1$ means, for 2 generations, $| < \bar{c}c - \bar{s}s > | \geq | < \bar{u}u + \bar{d}d > |$, which reasonably cannot be realized. However, for more generations, it may translate into the necessity for the condensate of the heaviest quark to become larger than that of light quarks, for example $| < \bar{t}t > | \geq | < \bar{u}u + \bar{d}d > |$. One must accordingly keep in mind that, to cure the (small) remaining discrepancy that may subsist for the masses of neutral pseudoscalars, one may have to invoke large condensates for heavy quarks, which can only eventually be achieved for more than 2 generations.

The mechanism likely to trigger such a large condensation remains of course to be uncovered. However, as we will show in the subsequent paper [18], some scalar(s) turn out to be strongly coupled to quarks, which we can identify as a trigger for the formation of a massive bound state (see subsection 7.3.2). The resulting breaking of the chiral symmetry can then go along with quark condensation in the appropriate channel. At this point, these are of course only conjectures.

5.12 Problems with leptonic decays $\pi^+ \to \ell^+\nu_\ell$ and $K^+ \to \ell^+\nu_\ell$ at $\theta_u = 0$

At this point, the situation looks globally satisfying, but for a few points that need to be clarified, concerning the orthogonality of a few mesons, the mass of the K^0 and the resulting $\hat b_X > 1$, and a paradoxical fermionic mixing angle. However this is illusory and we show below that the approximation $\theta_u = 0$ is completely at a loss to explain the leptonic decays of π^+ and K^+, which makes the situation worse than for 1 generation. Leptonic decays are the smoking gun of something very serious that we have missed.

5.12.1 No mixing at all : $\theta_d = 0 = \theta_u$

• $\pi^+ \to \ell^+\,\nu_\ell$ decay

π^+ occurs only in the X quadruplet, and one falls back on the same situation as in the case of 1 generation only (see section 3.8): the leptonic decays of charged pions are suitably described (of course up to the Cabibbo factor $\cos\theta_c$).

• $K^+ \to \ell^+\,\nu_\ell$ decay

We refer the reader to eq. (5.107) below for $\theta_d = 0$. While the amplitude is expected to vanish at $\theta_d = 0$, one finds it proportional to $\sqrt{b_\Omega(1-b_\Omega)}$. It is therefore unsuitably described, unless $b_\Omega \to 0$. We shall see that this only happens when both θ_d and θ_u are taken not vanishing.

5.12.2 $\theta_d \neq 0$ and setting $\theta_u = 0$: a lack of cancellation in leptonic decay amplitudes

• $\pi^+ \to \ell^+\,\nu_\ell$ decay

As soon as there is mixing, $\pi^+ \propto \bar u_m\gamma_5 d_m$ occurs in the 4 quadruplets X, H, Ω, Ξ such that

$$\pi^+ = \beta_X X^+ + \beta_H H^+ + \beta_\Omega \Omega^+ + \beta_\Xi \Xi^+. \tag{5.94}$$

One has accordingly for the looked for matrix element $\mathcal{M}_\pi$

$$\mathcal{M}_\pi \equiv \langle \ell\nu_\ell|\pi^+\rangle = \beta_X\langle \ell\nu_\ell|X^+\rangle + \beta_H\langle \ell\nu_\ell|H^+\rangle + \beta_\Omega\langle \ell\nu_\ell|\Omega^+\rangle + \beta_\Xi\langle \ell\nu_\ell|\Xi^+\rangle. \tag{5.95}$$

The leptonic decay of charged pions receive now a priori contributions from the 4 quadruplets X, H, Ω, Ξ. In the case of no mixing we noticed that the amplitude was controlled by $v_X/a_X \equiv v_X\beta_X$, and it is now controlled by the sum

$$v_X\beta_X + v_H\beta_H + v_\Omega\beta_\Omega + v_\Xi\beta_\Xi. \tag{5.96}$$

One needs therefore to know $\beta_X, \beta_H, \beta_\Omega, \beta_\Xi$. To that purpose we write

$$\begin{aligned}
X^+ &= a_X\pi^+ + f_X K^+ + c_X D^+ + d_X D_s^+,\\
H^+ &= a_H\pi^+ + f_H K^+ + c_H D^+ + d_H D_s^+,\\
\Omega^+ &= a_\Omega\pi^+ + f_\Omega K^+ + c_\Omega D^+ + d_\Omega D_s^+,\\
\Xi^+ &= a_\Xi\pi^+ + f_\Xi K^+ + c_\Xi D^+ + d_\Xi D_s^+.
\end{aligned} \tag{5.97}$$

Using PCAC and GMOR for π^+, PCAC for K^+, D^+, D_s^+, one gets

$$
\begin{aligned}
& a_X = c_d \frac{v_X}{f_\pi}, \quad a_H = 0, \quad a_\Omega = -a_X \frac{s_d}{c_d}\sqrt{\frac{b_\Omega}{b_X}}\frac{1}{r_\Omega}\frac{1}{\sqrt{2}}, \quad a_\Xi = -a_X \frac{s_d}{c_d}\sqrt{\frac{b_\Xi}{b_X}}\frac{1}{r_\Xi}\frac{1}{\sqrt{2}},\\
& f_X = a_X \frac{s_d}{c_d} \underbrace{\frac{f_K}{f_\pi}\frac{m_K^2}{m_\pi^2}\frac{m_u+m_d}{m_u+m_s}}_{F_K}, \quad f_H = 0,\\
& \qquad f_\Omega = a_X \sqrt{\frac{b_\Omega}{b_X}}\frac{1}{r_\Omega}\frac{1}{\sqrt{2}}\frac{f_K}{f_\pi}\frac{m_K^2}{m_\pi^2}\frac{m_u+m_d}{m_u+m_s}, \quad f_\Xi = a_X \sqrt{\frac{b_\Xi}{b_X}}\frac{1}{r_\Xi}\frac{1}{\sqrt{2}}\frac{f_K}{f_\pi}\frac{m_K^2}{m_\pi^2}\frac{m_u+m_d}{m_u+m_s},\\
& c_X = 0, \quad c_H = -a_X \frac{s_d}{c_d}\sqrt{\frac{b_H}{b_X}}\frac{1}{r_H}\frac{f_D}{f_\pi}\frac{m_D^2}{m_\pi^2}\frac{m_u+m_d}{m_c+m_d},\\
& \qquad c_\Omega = a_X \sqrt{\frac{b_\Omega}{b_X}}\frac{1}{r_\Omega}\frac{1}{\sqrt{2}}\frac{f_D}{f_\pi}\frac{m_D^2}{m_\pi^2}\frac{m_u+m_d}{m_c+m_d}, \quad c_\Xi = -a_X \sqrt{\frac{b_\Xi}{b_X}}\frac{1}{r_\Xi}\frac{1}{\sqrt{2}}\frac{f_D}{f_\pi}\frac{m_D^2}{m_\pi^2}\frac{m_u+m_d}{m_c+m_d},\\
& d_X = 0, \quad d_H = a_X \sqrt{\frac{b_H}{b_X}}\frac{1}{r_H}\frac{f_{D_s}}{f_\pi}\frac{m_{D_s}^2}{m_\pi^2}\frac{m_u+m_d}{m_c+m_s},\\
& \qquad d_\Omega = a_X \frac{s_d}{c_d}\sqrt{\frac{b_\Omega}{b_X}}\frac{1}{r_\Omega}\frac{1}{\sqrt{2}}\frac{f_{D_s}}{f_\pi}\frac{m_{D_s}^2}{m_\pi^2}\frac{m_u+m_d}{m_c+m_s}, \quad d_\Xi = -a_X \frac{s_d}{c_d}\sqrt{\frac{b_\Xi}{b_X}}\frac{1}{r_\Xi}\frac{1}{\sqrt{2}}\frac{f_{D_s}}{f_\pi}\frac{m_{D_s}^2}{m_\pi^2}\frac{m_u+m_d}{m_c+m_s}.
\end{aligned}
\tag{5.98}
$$

Because of (5.15), $\sqrt{\frac{b_H}{b_X}}\frac{1}{r_H} = \sqrt{\frac{1-b_H}{1-b_X}}, \quad \sqrt{\frac{b_\Omega}{b_X}}\frac{1}{r_\Omega} = \sqrt{\frac{1-b_\Omega}{1-b_X}}, \quad \sqrt{\frac{b_\Xi}{b_X}}\frac{1}{r_\Xi} = \sqrt{\frac{1-b_\Xi}{1-b_X}} \stackrel{b_\Xi=0}{=} \sqrt{\frac{1}{1-b_X}}$.

Because of (5.94) one must have

$$
\begin{aligned}
&\beta_X a_X + \beta_H a_H + \beta_\Omega a_\Omega + \beta_\Xi a_\Xi = 1,\\
&\beta_X f_X + \beta_H f_H + \beta_\Omega f_\Omega + \beta_\Xi f_\Xi = 0,\\
&\beta_X c_X + \beta_H c_H + \beta_\Omega c_\Omega + \beta_\Xi c_\Xi = 0,\\
&\beta_X d_X + \beta_H d_H + \beta_\Omega d_\Omega + \beta_\Xi d_\Xi = 0.
\end{aligned}
\tag{5.99}
$$

The solution of (5.99) and (5.98) is

$$
\begin{aligned}
&\beta_X = \frac{c_d^2}{a_X} = c_d \frac{f_\pi}{v_X}, \quad \beta_H = 0,\\
&\beta_\Omega = -\frac{1}{\sqrt{2}} c_d s_d \sqrt{\frac{1-b_\Omega}{1-b_X}}\frac{1}{a_X} = -\frac{1}{\sqrt{2}} s_d \sqrt{\frac{1-b_\Omega}{1-b_X}}\frac{f_\pi}{v_X},\\
&\beta_\Xi = -\frac{1}{\sqrt{2}} c_d s_d \sqrt{\frac{1-b_\Xi}{1-b_X}}\frac{1}{a_X} = -\frac{1}{\sqrt{2}} s_d \sqrt{\frac{1-b_\Xi}{1-b_X}}\frac{f_\pi}{v_X}.
\end{aligned}
\tag{5.100}
$$

According to (5.96) the leptonic decay amplitude of π^+ is accordingly proportional to

$$
f_\pi\; c_d \left(1 - \underbrace{\frac{s_d}{\sqrt{2}}\sqrt{\frac{b_\Omega(1-b_\Omega)}{b_X(1-b_X)}}}_{\approx 1}\right). \tag{5.101}
$$

Eq. (5.4) has been used to get rid of the contribution of Ξ. The numerical estimates uses the value of b_Ω which we had found in the case $\theta_u = 0$ (see section 5.10).

The nice agreement that took place with no mixing at all has disappeared when we only turn on the $d-s$ mixing. Another contribution is needed to cancel the one of Ω, which does not exist presently.

• $K^+ \to \ell^+ \nu_\ell$ decay

The equivalent of (5.94) is now

$$
K^+ = \zeta_X X^+ + \zeta_H H^+ + \zeta_\Omega \Omega^+ + \zeta_\Xi \Xi^+; \tag{5.102}
$$

that of (5.95) is

$$\mathcal{M}_K \equiv \langle \ell\nu_\ell | K^+\rangle = \zeta_X \langle \ell\nu_\ell | X^+\rangle + \zeta_H \langle \ell\nu_\ell | H^+\rangle + \zeta_\Omega \langle \ell\nu_\ell | \Omega^+\rangle + \zeta_\Xi \langle \ell\nu_\ell | \Xi^+\rangle, \tag{5.103}$$

which is now proportional to (the other terms are trivial factors)

$$v_X \zeta_X + v_H \zeta_H + v_\Omega \zeta_\Omega + v_\Xi \zeta_\Xi. \tag{5.104}$$

The system (5.99) is replaced with

$$\begin{aligned} &\zeta_X a_X + \zeta_H a_H + \zeta_\Omega a_\Omega + \zeta_\Xi a_\Xi = 0,\\ &\zeta_X f_X + \zeta_H f_H + \zeta_\Omega f_\Omega + \zeta_\Xi f_\Xi = 1,\\ &\zeta_X c_X + \zeta_H c_H + \zeta_\Omega c_\Omega + \zeta_\Xi c_\Xi = 0,\\ &\zeta_X d_X + \zeta_H d_H + \zeta_\Omega d_\Omega + \zeta_\Xi d_\Xi = 0. \end{aligned} \tag{5.105}$$

Combined with (5.98) it leads to

$$\begin{aligned} &\zeta_X = s_d c_d \frac{1}{a_X}\frac{1}{F_K} = s_d \frac{1}{F_K}\frac{f_\pi}{v_X}, \quad \zeta_H = 0,\\ &\zeta_\Omega = \frac{1}{\sqrt{2}} c_d^2 \sqrt{\frac{1-b_\Omega}{1-b_X}}\frac{1}{a_X}\frac{1}{F_K} = \frac{1}{\sqrt{2}} c_d \sqrt{\frac{1-b_\Omega}{1-b_X}}\frac{1}{F_K}\frac{f_\pi}{v_X},\\ &\zeta_\Xi = \frac{1}{\sqrt{2}} c_d^2 \sqrt{\frac{1-b_\Xi}{1-b_X}}\frac{1}{a_X}\frac{1}{F_K} = \frac{1}{\sqrt{2}} c_d \sqrt{\frac{1-b_\Xi}{1-b_X}}\frac{1}{F_K}\frac{f_\pi}{v_X},\\ &with\ F_K = \frac{f_K}{f_\pi}\frac{m_K^2}{m_\pi^2}\frac{m_u+m_d}{m_u+m_s}. \end{aligned} \tag{5.106}$$

Like for pion leptonic decays, $b_\Xi = 0$ ensures the vanishing of the Ξ contribution, such that, by (5.104), the amplitude gets controlled by

$$v_X \zeta_X + v_\Omega \zeta_\Omega = \frac{f_\pi}{F_K}\ s_d \left(1 + \frac{1}{\sqrt{2}}\frac{c_d}{s_d}\sqrt{\frac{b_\Omega(1-b_\Omega)}{b_X(1-b_X)}}\right). \tag{5.107}$$

Like for charged pions, the fair agreement that would be obtained from the contribution of X^+ alone gets totally spoiled by that from Ω^+.

Note: the parameter f_π/F_K that occurs in (5.107) also writes

$$\frac{f_\pi}{F_K} \approx f_K \frac{<\bar u u + \bar d d>}{<\bar u u + \bar s s>}. \tag{5.108}$$

5.13 Conclusion for the case $\theta_d \neq 0, \theta_u = 0$

The approximation of setting $\theta_u = 0$ has given a fairly good estimate of θ_d, and a matching with the physics of pseudoscalar mesons much better than could have been anticipated. However, dark points subsist than we enumerate below.

* we find too large values of $b_\Omega, \hat b_\Omega$ and of the non-diagonal quark condensates $<\bar u c>, <\bar c u>$;

* the mass of the neutral kaon is found too small by $140\,MeV$ unless one goes to $\hat b_X > 1$; however, this value conflicts with strongly desired orthogonality relations;

* the value of the $d-s$ mixing angle that we find from bosonic argumentation practically coincides with a pole of the ratio of fermionic mass terms $\frac{\mu_{ds}}{\mu_d-\mu_s}$, which would corresponds to a maximal “fermionic” mixing; this paradox could be avoided if b_Ω and $\hat b_\Omega$ were very small, which does not occur at $\theta_u = 0$;

* leptonic decays of π^+ and K^+ are totally off, unless one also switches off θ_d or if $b_\Omega, \hat b_\Omega \to 0$.

Chapter 6

2 generations with $\theta_d \neq 0, \theta_u \neq 0$

We show in this chapter that the situation largely improves by taking $\theta_u \neq 0$ and proceed as follows.
* $\theta_d - \theta_u = \theta_c$ will be taken to its experimental value (1.6);
* θ_u will be taken to satisfy (4.21);
* b_X, b_H, b_Ω are expressed according to (4.20) as functions of θ_u, θ_d and δ;
* that leptonic decays of π^+ and K^+ are suitably described provides a relation between θ_u, b_H and b_Ω which determines the value of δ;
* the masses of π^0, K^0 and D^0 are then used to determine $\hat{b}_X$, $\hat{b}_\Omega$ and $\hat{r}_H$.

The masses of neutral pseudoscalar mesons are now fairly well described with $\hat{b}_X < 1$. b_Ω (and probably $\hat{b}_\Omega$, too) become very small, which presumably solves the paradox of bosonic versus fermionic mixing. At the opposite, b_H departs from its previously very small value. Leptonic decays of charged pions and kaons are correctly accounted for.

6.1 An estimate of θ_u

On Fig.6.1 we plot in blue θ_d as a function of θ_u as given by (4.21), and we also plot in purple the corresponding value of $\theta_d - \theta_u$ that we identify with the "Cabibbo angle" of the GSW model. This last curve crosses the physical value (1.6) drawn in yellow at

$$\boxed{\theta_u \approx .04225 \ll \theta_d} \tag{6.1}$$

A peculiarity of θ_u as given by (4.21) as a function of θ_d

$$\theta_u = \arccos\left(\frac{1}{\sqrt{2}}\sqrt{1 + \frac{1+a}{1-a}\cos 2\theta_d}\right), \quad a = \frac{\frac{1}{m^2_{K^\pm}} - \frac{1}{m^2_{D^\pm}}}{\frac{1}{m^2_{\pi^\pm}} - \frac{1}{m^2_{D_s^\pm}}}. \tag{6.2}$$

is that:
* it is, as shown on Fig. 6.2 (blue curve), a rather rapidly varying function of θ_d;
* it has no reliable expansion at $\theta_d \to 0$ (since (4.21) has no solution); if one brutally perform a formal expansion of θ_u in powers of θ_d one gets

$$\theta_u \stackrel{\theta_d \to 0}{\longrightarrow} \arccos\frac{1}{\sqrt{1-a}} + \frac{1+a}{2\sqrt{1-a}\sqrt{-\frac{a}{1-a}}}\,\theta_d^2 + \ldots \tag{6.3}$$

which is clearly meaningless since $a < 1$ yields a second term that is imaginary;
* its expansion in powers of $a \simeq \frac{m_\pi^2}{m_K^2} \ll 1$ starts with

$$\theta_u \stackrel{a \to 0}{\longrightarrow} \theta_d - \frac{a}{\tan 2\theta_d} - \frac{a^2}{\tan 2\theta_d}\frac{1}{\sin^2 2\theta_d} + \ldots \tag{6.4}$$

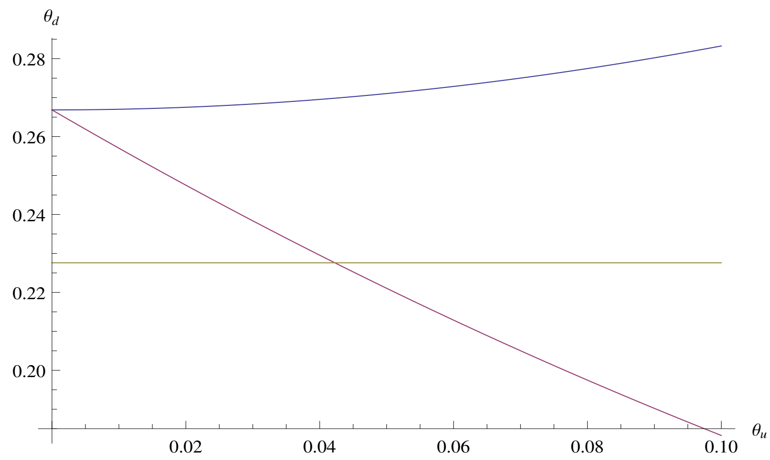

Figure 6.1: θ_d (blue) and $\theta_d - \theta_u$ (purple) as functions of θ_u, compared with the experimental Cabibbo value $\theta_c \approx .22759$ (yellow)

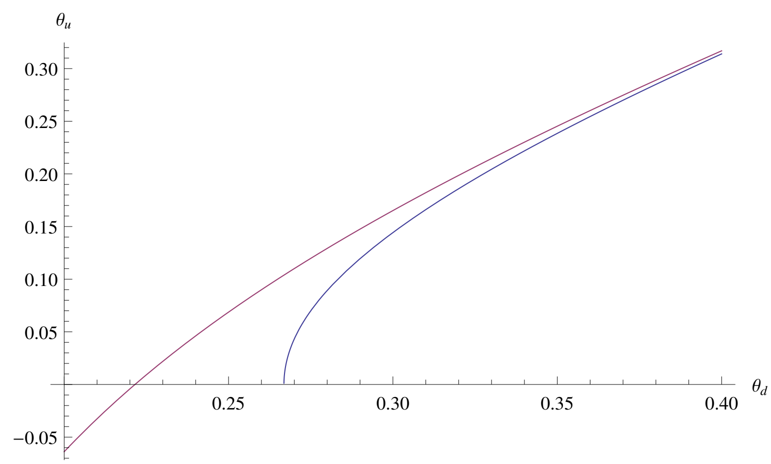

Figure 6.2: θ_u as a function of θ_d (blue curve); its expansion at 2nd order in a, parameter given in (6.2) (purple curve)

in which the limit $\theta_d \to 0$ must coincide with $a \to 0$, that is $m_\pi \to 0$.

The purple curve on Fig.6.2 corresponds to the first 2 terms of the expansion (6.4). It shows that there are quantities, in particular the ones that depend on θ_u, for which θ_d cannot be straightforwardly considered as a small number.

6.2 Leptonic decays of pions and kaons

Since $\theta_u \to 0$ has been seen in the previous chapter to lead to erroneous leptonic decays of charged pions and kaons, we start our new investigations with these decays.

For $\theta_u \neq 0$ and $\theta_d \neq 0$, eqs. (5.98) are replaced with

$$
\begin{aligned}
& a_X = c_u c_d \frac{v_X}{f_\pi}, \quad a_H = a_X \frac{s_u s_d}{c_u c_d}\sqrt{\frac{b_H}{b_X}}\frac{1}{r_H}, \quad a_\Omega = a_X(-)\frac{s_{u+d}}{c_u c_d}\sqrt{\frac{b_\Omega}{b_X}}\frac{1}{r_\Omega}\frac{1}{\sqrt{2}}, \quad a_\Xi = a_X \frac{s_{u-d}}{c_u c_d}\sqrt{\frac{b_\Xi}{b_X}}\frac{1}{r_\Xi}\frac{1}{\sqrt{2}},\\
& f_X = a_X \frac{s_d}{c_d}\underbrace{\frac{f_K}{f_\pi}\frac{m_K^2}{m_\pi^2}\frac{m_u+m_d}{m_u+m_s}}_{F_K}, \quad f_H = a_X(-)\frac{s_u}{c_u}\sqrt{\frac{b_H}{b_X}}\frac{1}{r_H}\frac{f_K}{f_\pi}\frac{m_K^2}{m_\pi^2}\frac{m_u+m_d}{m_u+m_s},\\
& \qquad f_\Omega = a_X \frac{c_{u+d}}{c_u c_d}\sqrt{\frac{b_\Omega}{b_X}}\frac{1}{r_\Omega}\frac{1}{\sqrt{2}}\frac{f_K}{f_\pi}\frac{m_K^2}{m_\pi^2}\frac{m_u+m_d}{m_u+m_s}, \quad f_\Xi = a_X \frac{c_{u-d}}{c_u c_d}\sqrt{\frac{b_\Xi}{b_X}}\frac{1}{r_\Xi}\frac{1}{\sqrt{2}}\frac{f_K}{f_\pi}\frac{m_K^2}{m_\pi^2}\frac{m_u+m_d}{m_u+m_s},\\
& c_X = a_X \frac{s_u}{c_u}\frac{f_D}{f_\pi}\frac{m_D^2}{m_\pi^2}\frac{m_u+m_d}{m_c+m_d}, \quad c_H = a_X(-)\frac{s_d}{c_d}\sqrt{\frac{b_H}{b_X}}\frac{1}{r_H}\frac{f_D}{f_\pi}\frac{m_D^2}{m_\pi^2}\frac{m_u+m_d}{m_c+m_d},\\
& \qquad c_\Omega = a_X \frac{c_{u+d}}{c_u c_d}\sqrt{\frac{b_\Omega}{b_X}}\frac{1}{r_\Omega}\frac{1}{\sqrt{2}}\frac{f_D}{f_\pi}\frac{m_D^2}{m_\pi^2}\frac{m_u+m_d}{m_c+m_d}, \quad c_\Xi = a_X(-)\frac{c_{u-d}}{c_u c_d}\sqrt{\frac{b_\Xi}{b_X}}\frac{1}{r_\Xi}\frac{1}{\sqrt{2}}\frac{f_D}{f_\pi}\frac{m_D^2}{m_\pi^2}\frac{m_u+m_d}{m_c+m_d},\\
& d_X = a_X \frac{s_u s_d}{c_u c_d}\frac{f_{D_s}}{f_\pi}\frac{m_{D_s}^2}{m_\pi^2}\frac{m_u+m_d}{m_c+m_s}, \quad d_H = a_X\sqrt{\frac{b_H}{b_X}}\frac{1}{r_H}\frac{f_{D_s}}{f_\pi}\frac{m_{D_s}^2}{m_\pi^2}\frac{m_u+m_d}{m_c+m_s},\\
& \qquad d_\Omega = a_X \frac{s_{u+d}}{c_u c_d}\sqrt{\frac{b_\Omega}{b_X}}\frac{1}{r_\Omega}\frac{1}{\sqrt{2}}\frac{f_{D_s}}{f_\pi}\frac{m_{D_s}^2}{m_\pi^2}\frac{m_u+m_d}{m_c+m_s}, \quad d_\Xi = a_X \frac{s_{u-d}}{c_u c_d}\sqrt{\frac{b_\Xi}{b_X}}\frac{1}{r_\Xi}\frac{1}{\sqrt{2}}\frac{f_{D_s}}{f_\pi}\frac{m_{D_s}^2}{m_\pi^2}\frac{m_u+m_d}{m_c+m_s}.
\end{aligned}
\tag{6.5}
$$

6.2.1 $\pi^+ \to \ell^+\nu_\ell$

Combining (6.5) and (5.99) yields

$$
\begin{aligned}
\beta_X &= \frac{c_u^2 c_d^2}{a_X} = c_u c_d \frac{f_\pi}{v_X},\\
\beta_H &= \frac{c_u c_d s_u s_d}{a_X}\sqrt{\frac{1-b_H}{1-b_X}} = s_u s_d \frac{f_\pi}{v_X}\sqrt{\frac{1-b_H}{1-b_X}},\\
\beta_\Omega &= -\frac{1}{\sqrt{2}}\frac{s_{u+d} c_u c_d}{a_X}\sqrt{\frac{1-b_\Omega}{1-b_X}} = -\frac{1}{\sqrt{2}}s_{u+d}\frac{f_\pi}{v_X}\sqrt{\frac{1-b_\Omega}{1-b_X}},\\
\beta_\Xi &= \frac{1}{\sqrt{2}}\frac{s_{u-d} c_u c_d}{a_X}\sqrt{\frac{1-b_\Xi}{1-b_X}} = \frac{1}{\sqrt{2}}s_{u-d}\frac{f_\pi}{v_X}\sqrt{\frac{1-b_\Xi}{1-b_X}},
\end{aligned}
\tag{6.6}
$$

which replaces (5.100). According to (5.96) and owing to $b_\Xi = 0$, this makes the corresponding amplitude controlled by

$$
f_\pi\ c_d \left(c_u + s_u \frac{s_d}{c_d}\sqrt{\frac{b_H(1-b_H)}{b_X(1-b_X)}} - \frac{1}{\sqrt{2}}\frac{s_{u+d}}{c_d}\sqrt{\frac{b_\Omega(1-b_\Omega)}{b_X(1-b_X)}}\right). \tag{6.7}
$$

6.2.2 $K^+ \to \ell^+\nu_\ell$

Combining (6.5) and (5.105) gives

$$
\begin{aligned}
\zeta_X &= \frac{c_u^2 s_d c_d}{a_X F_K} = \frac{c_u s_d}{F_K}\frac{f_\pi}{v_X},\\
\zeta_H &= -\frac{s_u c_u c_d^2}{a_X F_K}\sqrt{\frac{1-b_H}{1-b_X}} = -\frac{s_u c_d}{F_K}\sqrt{\frac{1-b_H}{1-b_X}}\frac{f_\pi}{v_X},\\
\zeta_\Omega &= \frac{1}{\sqrt{2}}\frac{c_{u+d} c_u c_d}{a_X F_K}\sqrt{\frac{1-b_\Omega}{1-b_X}} = \frac{1}{\sqrt{2}}\frac{c_{u+d}}{F_K}\sqrt{\frac{1-b_\Omega}{1-b_X}}\frac{f_\pi}{v_X},\\
\zeta_\Xi &= \frac{1}{\sqrt{2}}\frac{c_{u-d} c_u c_d}{a_X F_K}\sqrt{\frac{1-b_\Xi}{1-b_X}} = \frac{1}{\sqrt{2}}\frac{c_{u-d}}{F_K}\sqrt{\frac{1-b_\Xi}{1-b_X}}\frac{f_\pi}{v_X},
\end{aligned}
\tag{6.8}
$$

which replaces (5.106). According to (5.104) the leptonic decay amplitude of K^+ is now controlled by

$$
\frac{f_\pi}{F_K}\ s_d \left(c_u - s_u \frac{c_d}{s_d}\sqrt{\frac{b_H(1-b_H)}{b_X(1-b_X)}} + \frac{1}{\sqrt{2}}\frac{c_{u+d}}{s_d}\sqrt{\frac{b_\Omega(1-b_\Omega)}{b_X(1-b_X)}}\right). \tag{6.9}
$$

We recall that f_π/F_K is given in (5.108).

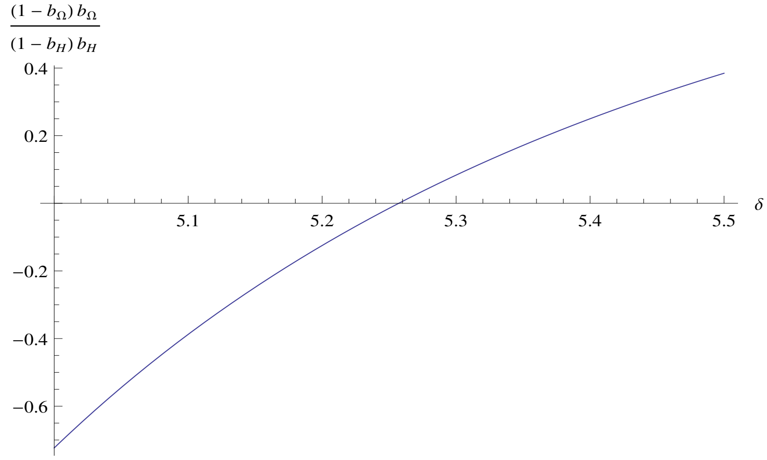

Figure 6.3: $\frac{b_\Omega(1-b_\Omega)}{b_H(1-b_H)}$ as a function of δ

6.2.3 Updating $b_X, b_H, b_\Omega, \delta$

Since the first term in (6.7) and (6.9) gives a fair description of leptonic decays of charged pions and kaons, a cancellation between the other 2 contributions is wished for. For $\theta_u \ll \theta_d$, this requires, for both pions and kaons

$$\theta_u \approx \frac{1}{\sqrt{2}}\sqrt{\frac{b_\Omega(1-b_\Omega)}{b_H(1-b_H)}}. \tag{6.10}$$

The values (see (5.81)) that we obtained at $\theta_u = 0$ for the parameters b_H and b_Ω appear now grossly erroneous since they yield $\theta_u \approx 4.18$.

Since $\theta_u \ll 1$ (see (6.1)), (6.10) points out at $b_\Omega \approx 0$ or $b_\Omega \approx 1$. However, the b's being themselves functions of θ_d and θ_u (see (6.11), (6.12), (6.13) below), this cannot be settled without, in particular, re-investigating the masses of pseudoscalar mesons, for $\theta_u \neq 0$.

Using r_1, r_2, r_3, r_4 defined in (5.6), one gets

$$b_\Omega = 1 - \frac{c_{2(u+d)} + c_{2u}c_{2d}}{-(c_{2(u-d)} + c_{2u}c_{2d}) + \delta\,(c_{2u}c_{2d}\,r_1 - r_4)}, \tag{6.11}$$

$$b_X = 1 - \frac{2}{\dfrac{\delta r_2}{c_{2d}} + \dfrac{\delta}{2}\left(r_1 + \dfrac{r_4}{c_{2u}c_{2d}}\right) - \dfrac{1}{2}\dfrac{1}{1-b_\Omega}\left(1 - \dfrac{c_{2(u+d)}}{c_{2u}c_{2d}}\right) - \dfrac{1}{2}\left(1 - \dfrac{c_{2(u-d)}}{c_{2u}c_{2d}}\right)}, \tag{6.12}$$

$$b_H = 1 - \frac{2}{-\dfrac{\delta r_2}{c_{2d}} + \dfrac{\delta}{2}\left(r_1 + \dfrac{r_4}{c_{2u}c_{2d}}\right) - \dfrac{1}{2}\dfrac{1}{1-b_\Omega}\left(1 - \dfrac{c_{2(u+d)}}{c_{2u}c_{2d}}\right) - \dfrac{1}{2}\left(1 - \dfrac{c_{2(u-d)}}{c_{2u}c_{2d}}\right)}. \tag{6.13}$$

We shall take the value (6.1) of θ_u and the experimental value (1.6) for $\theta_d - \theta_u$. Then, (6.10) determines the value of δ. In Fig. 6.3 we plot the square of the r.h.s. of (6.10) as a function of δ. Because $2\theta_u^2 \approx .0032 \ll 1$, the solution is very close to the value at which the blue curve in Fig.6.3 crosses the horizontal axis. One gets

$$\boxed{\delta \approx 5.259\,GeV^2} \tag{6.14}$$

It also corresponds, in agreement with our intuition, to $b_\Omega \ll 1$, more precisely, from (6.11), putting in the physical values of the charged pseudoscalar mesons masses

$$\boxed{b_\Omega \approx 7\,10^{-4}} \tag{6.15}$$

This limit $b_\Omega \to 0$, that we have only been able to achieve at $\theta_u \neq 0$, turns out to also correspond to suitable leptonic decays of charged pions and kaons when θ_u was taken to 0 (see (5.101) and (5.107)). It however could not be justified, then.

One also gets, respectively from (6.12) and (6.11)

$$\boxed{b_X \approx .996564, \quad b_H \approx .27743} \tag{6.16}$$

6.2.4 The small value of b_Ω

Like b_H at $\theta_u = 0$ (see subsection 5.10.1), b_Ω given in (6.11) does not have either a reliable expansion at the chiral limit $m_\pi \to 0$; it starts indeed with

$$b_\Omega \stackrel{m_\pi \to 0}{\simeq} 1 - \frac{c_{2(d-u)} + 3\, c_{2(d+u)}}{\delta(-2 + c_{2(d-u)} + c_{2(d+u)})}\, m_\pi^2 + \ldots \tag{6.17}$$

in which the term between parentheses in the denominator is small. The best that we can do, like we did for b_H, is to write $\delta = \delta_{b_\Omega=0} + \zeta m_\pi^2$, in which $\delta_{b_\Omega=0} = \dfrac{4c_{2u}c_{2d}}{c_{2u}c_{2d}r_1 - r_4}$ is the value of δ at which $b_\Omega = 0$. (6.11) then yields $b_\Omega \simeq \zeta m_\pi^2 \left(\frac{1}{m_K^2} + \frac{1}{m_D^2}\right)$. Numerically, ζ is not vanishing because θ_u is not either and, numerically, one gets $\zeta = \dfrac{\delta - \delta_{b_\Omega=0}}{m_\pi^2} \approx .093$, such that

$$b_\Omega \approx .093\, m_\pi^2 \left(\frac{1}{m_K^2} + \frac{1}{m_D^2}\right). \tag{6.18}$$

More comments concerning b_Ω will be made in subsection 6.6.2.

6.3 Neutral pseudoscalar mesons

The relations that come out of the orthogonality conditions among neutral pseudoscalars turn out to be the same as for $\theta_u = 0$; like before, the η meson fails to be orthogonal to $K^0 + \bar K^0$, and also now to $D^0 + \bar D^0$.
The masses of π^0, K^0 and D^0 get now in good agreement with experiment without invoking $\hat b_X > 1$.

6.3.1 Orthogonality

We refer to the general eqs. (4.22). To solve them, we use the results (4.17), which are valid whatever θ_u and θ_d.

• Use $(a), (c)$ (we recall $\hat b_H = 1 \Rightarrow \hat\delta_H = 0$).

$$(a) - (c) \Rightarrow \frac{\delta_\Xi}{\nu_\Xi^4} = \frac{\hat\delta_\Xi}{\hat\nu_\Xi^4}, \tag{6.19}$$

$$(a) + (c) \Rightarrow -c_{2(u-d)}\frac{\delta_\Omega}{\nu_\Omega^4} + \frac{1}{2}s_{2u}s_{2d}\frac{\hat\delta_X}{\hat\nu_X^4} + c_{2u}c_{2d}\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4} = 0. \tag{6.20}$$

• Use $(i), (j)$.

$$\begin{aligned} (i) &\Rightarrow \frac{1}{2}s_{2u}^2\frac{\hat\delta_X}{\hat\nu_X^4} + c_{2u}^2\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4} - \frac{\hat\delta_\Xi}{\hat\nu_\Xi^4} = 0,\\ (j) &\Rightarrow +\frac{1}{2}s_{2d}^2\frac{\hat\delta_X}{\hat\nu_X^4} + c_{2d}^2\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4} - \frac{\hat\delta_\Xi}{\hat\nu_\Xi^4} = 0. \end{aligned} \tag{6.21}$$

• Using $(i), (j)$ and $(a) - (c)$ yields

$$\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4} = \frac{\delta_\Omega}{\nu_\Omega^4}, \tag{6.22}$$

and

$$\frac{\hat\delta_X}{\hat\nu_X^4} = 2\frac{\delta_\Omega}{\nu_\Omega^4}. \tag{6.23}$$

Then, $(a)+(c)$ is satisfied.

• The results (6.19), (6.22), (6.23), together with (4.17) combine accordingly into

$$\boxed{\frac{b_X(1-b_X)}{\mu_X^6} = \frac{b_H(1-b_H)}{\mu_H^6} = \frac{b_\Omega(1-b_\Omega)}{\mu_\Omega^6} = \frac{1}{2}\frac{\hat b_X(1-\hat b_X)}{\hat\mu_X^6} = \frac{\hat b_\Omega(1-\hat b_\Omega)}{\hat\mu_\Omega^6}} \tag{6.24}$$

like for $\theta_u = 0$ (see (5.14) and (5.19)).

• while (e) and (g) are verified while (f) and (h) are not: like for $\theta_u = 0$, π^0 is orthogonal to $K^0+\bar K^0$ and $D^0+\bar D^0$ but η is not. We check below that, indeed, neither (g) and (h), nor (e) and (f) can be simultaneously satisfied.

* (g) and (h), which correspond respectively to the orthogonality of π^0 and η [1] to $K^0+\bar K^0$, cannot be satisfied simultaneously.

$$\begin{aligned}
&(g)+(h): -c_u^2 s_d c_d\frac{\delta_X}{\nu_X^4} + c_u^2 s_d c_d\frac{\hat\delta_X}{\hat\nu_X^4} + s_u^2 s_d c_d\frac{\delta_H}{\nu_H^4} - s_u^2 s_d c_d\overbrace{\frac{\hat\delta_H}{\hat\nu_H^4}}^{0} + \frac{1}{2}s_{2u}c_{2d}\frac{\delta_\Omega}{\nu_\Omega^4} - \frac{1}{2}s_{2u}c_{2d}\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4} = 0\\
&\qquad\Rightarrow s_{2(u-d)}\frac{\delta_\Omega}{\nu_\Omega^4} + c_u^2 s_{2d}\frac{\hat\delta_X}{\hat\nu_X^4} - s_{2u}c_{2d}\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4} = 0;\\
&(g)-(h): -c_d^2 s_d c_d\frac{\delta_X}{\nu_X^4} - c_d^2 s_d c_d\frac{\hat\delta_X}{\hat\nu_X^4} + s_d^2 s_d c_d\frac{\delta_H}{\nu_H^4} + s_d^2 s_d c_d\overbrace{\frac{\hat\delta_H}{\hat\nu_H^4}}^{0} + \frac{1}{2}s_{2d}c_{2d}\frac{\delta_\Omega}{\nu_\Omega^4} + \frac{1}{2}s_{2d}c_{2d}\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4} = 0\\
&\qquad\Rightarrow c_{2d}\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4} = c_d^2\frac{\hat\delta_X}{\hat\nu_X^4}.
\end{aligned} \tag{6.25}$$

which entails

$$\frac{\hat\delta_X}{\hat\nu_X^4} = \frac{1}{c_u c_d}\frac{s_{2(u-d)}}{2s_{u-d}}\,\frac{\delta_\Omega}{\nu_\Omega^4}, \tag{6.26}$$

different from (6.23), and

$$\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4} = \frac{c_d}{c_u c_{2d}}\frac{s_{2(u-d)}}{2s_{u-d}}\,\frac{\delta_\Omega}{\nu_\Omega^4}, \tag{6.27}$$

different from (6.22).

* (e) and (f), which correspond to the orthogonality of π^0 and η to $D^0+\bar D^0$, cannot be satisfied simultaneously either.

$$\begin{aligned}
&(e)+(f) \Rightarrow s_{2u}(c_u^2\frac{\hat\delta_X}{\hat\nu_X^4} - c_{2u}\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4}) = 0,\\
&(e)-(f) \Rightarrow \frac{\delta_\Omega}{\nu_\Omega^4}(c_{2d}s_{2u} - s_{2d}c_{2u}) = \frac{\hat\delta_X}{\hat\nu_X^4}s_{2u}c_d^2 - \frac{\hat\delta_\Omega}{\hat\nu_\Omega^4}s_{2d}c_{2u}.
\end{aligned} \tag{6.28}$$

This yields

$$\frac{\hat\delta_X}{\hat\nu_X^4} = \frac{c_{2d}s_{2u} - s_{2d}c_{2u}}{s_{2u}c_d^2 - s_{2d}c_u^2}\,\frac{\delta_\Omega}{\nu_\Omega^4}, \tag{6.29}$$

different from both (6.23) and (6.26) and

$$\frac{\hat\delta_\Omega}{\hat\nu_\Omega^4} = \frac{c_u^2}{c_{2u}}\,\frac{c_{2d}s_{2u} - s_{2d}c_{2u}}{s_{2u}c_d^2 - s_{2d}c_u^2}\,\frac{\delta_\Omega}{\nu_\Omega^4}, \tag{6.30}$$

different from both (6.22) and (6.27)

[1] The interpolating fields of which being defined, as before, as being proportional respectively to $\bar u\gamma_5 u - \bar d\gamma_5 d$ and $\bar u\gamma_5 u + \bar d\gamma_5 d$

6.3.2 Masses of π^0, K^0, D^0

We refer to eqs. (4.23), (4.24) and (4.25). To get to the following expressions we use the relations (4.17) and the following definitions and tricks.

$$
\begin{aligned}
&\frac{\delta_i}{\nu_i^4} = \delta\hat{v}_H^2 \frac{b_i(1-b_i)}{2\mu_i^6},\\
&\hat{b}_H = 1 \Rightarrow \hat{\delta}_H = 0,\\
&\frac{b_X(1-b_X)}{\mu_X^6} = \frac{b_H(1-b_H)}{\mu_H^6} = \frac{b_\Omega(1-b_\Omega)}{\mu_\Omega^6} = \frac{b_\Xi(1-b_\Xi)}{\mu_\Xi^6} = \frac{1}{2}\frac{\hat{b}_X(1-\hat{b}_X)}{\hat{\mu}_X^6} = \frac{\hat{b}_\Omega(1-\hat{b}_\Omega)}{\hat{\mu}_\Omega^6} = \frac{\hat{b}_\Xi(1-\hat{b}_\Xi)}{\hat{\nu}_\Xi^6},\\
&\frac{1}{\nu_i^4} \equiv \hat{v}_H^2 \frac{b_i}{2\mu_i^6} = \hat{v}_H^2 \frac{1}{1-b_i}\,\frac{b_i(1-b_i)}{2\mu_i^6},\\
&\frac{1}{\hat{\nu}_H^4} \equiv \hat{v}_H^2 \frac{\hat{b}_H}{2\hat{\mu}_H^6} = \hat{v}_H^2 \frac{1}{2\mu_X^6}\underbrace{\frac{\mu_X^6}{\hat{\mu}_H^6}}_{1/\hat{r}_H^2} = \hat{v}_H^2 \frac{1}{b_X(1-b_X)}\frac{b_X(1-b_X)}{2\mu_X^6}\frac{1}{\hat{r}_H^2},\\
&b_\Xi = 0 = \hat{b}_\Xi \Rightarrow \frac{1}{1-b_\Xi} = 1 = \frac{1}{1-\hat{b}_\Xi},
\end{aligned}
\tag{6.31}
$$

and one cancels $\hat{v}_H^2$ between numerators and denominators. This gives

$$
\begin{aligned}
m_{\pi^0}^2 &= \delta\,\frac{(c_u^2+c_d^2)^2 + 2(c_u^2-c_d^2)^2 + (s_u^2+s_d^2)^2 + s_{2u}^2 + s_{2d}^2}{\frac{(c_u^2+c_d^2)^2}{1-b_X} + 2\frac{(c_u^2-c_d^2)^2}{1-\hat{b}_X} + \frac{(s_u^2+s_d^2)^2}{1-b_H} + \frac{(s_u^2-s_d^2)^2}{b_X(1-b_X)}\frac{1}{\hat{r}_H^2} + \frac{1}{2}\frac{(s_{2u}+s_{2d})^2}{1-b_\Omega} + \frac{1}{2}\frac{(s_{2u}-s_{2d})^2}{1-\hat{b}_\Omega}}\\
&\approx \frac{4\delta}{\frac{(c_u^2+c_d^2)^2}{1-b_X} + 2\frac{(c_u^2-c_d^2)^2}{1-\hat{b}_X} + \frac{(s_u^2+s_d^2)^2}{1-b_H} + \frac{(s_u^2-s_d^2)^2}{b_X(1-b_X)}\frac{1}{\hat{r}_H^2} + \frac{1}{2}\frac{(s_{2u}+s_{2d})^2}{1-b_\Omega} + \frac{1}{2}\frac{(s_{2u}-s_{2d})^2}{1-\hat{b}_\Omega}},
\end{aligned}
\tag{6.32}
$$

$$
\begin{aligned}
m_{K^0}^2 &= \delta\,\frac{\overbrace{2s_d^2c_d^2 + \frac{1}{2}c_{2d}^2 + \frac{1}{2}}^{1}}{\frac{s_d^2c_d^2}{2}\left(\frac{1}{1-b_X} + 2\frac{1}{1-\hat{b}_X} + \frac{1}{1-b_H} + \frac{1}{b_X(1-b_X)}\frac{1}{\hat{r}_H^2}\right) + \frac{c_{2d}^2}{4}\left(\frac{1}{1-b_\Omega} + \frac{1}{1-\hat{b}_\Omega}\right) + \frac{1}{2}}\\
&= \frac{4\delta}{\frac{s_{2d}^2}{2}\left(\frac{1}{1-b_X} + 2\frac{1}{1-\hat{b}_X} + \frac{1}{1-b_H} + \frac{1}{b_X(1-b_X)}\frac{1}{\hat{r}_H^2}\right) + c_{2d}^2\left(\frac{1}{1-b_\Omega} + \frac{1}{1-\hat{b}_\Omega}\right) + 2},
\end{aligned}
\tag{6.33}
$$

$$
\begin{aligned}
m_{D^0}^2 &= \delta\,\frac{\overbrace{2s_u^2c_u^2 + \frac{1}{2}c_{2u}^2 + \frac{1}{2}}^{1}}{\frac{s_u^2c_u^2}{2}\left(\frac{1}{1-b_X} + 2\frac{1}{1-\hat{b}_X} + \frac{1}{1-b_H} + \frac{1}{b_X(1-b_X)}\frac{1}{\hat{r}_H^2}\right) + \frac{c_{2u}^2}{4}\left(\frac{1}{1-b_\Omega} + \frac{1}{1-\hat{b}_\Omega}\right) + \frac{1}{2}}\\
&= \frac{4\delta}{\frac{s_{2u}^2}{2}\left(\frac{1}{1-b_X} + 2\frac{1}{1-\hat{b}_X} + \frac{1}{1-b_H} + \frac{1}{b_X(1-b_X)}\frac{1}{\hat{r}_H^2}\right) + c_{2u}^2\left(\frac{1}{1-b_\Omega} + \frac{1}{1-\hat{b}_\Omega}\right) + 2}.
\end{aligned}
\tag{6.34}
$$

Inside (6.32), (6.33) and (6.34), b_Ω, b_X, b_H are given by (6.11), (6.12) and (6.13) in terms of $\delta, \theta_d, \theta_u$ and of the masses of the charged pseudoscalar mesons. Since $\theta_c \equiv \theta_d - \theta_u$ and θ_u are respectively given by (1.6) and (6.1), δ by (6.14), b_H and b_X by (6.13) and (6.12), the 3 equations (6.32), (6.33) and (6.34) should be enough to determine

$\hat b_X, \hat b_\Omega, \hat r_H$. Numerical studies show that:
* the masses of π^0 [2] and K^0 can be suitably accounted for at the condition that

$$\boxed{\hat r_H \equiv \frac{<\bar c c - \bar s s>}{<\bar u u + \bar d d>} \geq .945} \tag{6.35}$$

They are practically insensitive to the value of $\hat b_\Omega$ which, supposedly small, like b_Ω, according to our intuition, can even be considered to be vanishing;
* the mass of D^0 is only off by $\approx 20\,MeV$;
* $\hat b_X$ should be $\mathcal{O}(1)$ but, unfortunately, one gets too small a sensitivity to determine this parameter accurately; a more exhaustive study of the full system of equations is probably necessary for this.

To give an idea of the precision of the determination, for $\hat r_H = .96, \hat b_\Omega = 0$ and $\hat b_X = .8$, one gets

$$\begin{aligned}
m_{\pi^0} &\approx 139.38\,MeV = m_{\pi^+}^{exp} - 190\,KeV = m_{\pi^0}^{exp} + 4.42\,MeV,\\
m_{K^0} &\approx 496.8\,MeV = m_{K^0}^{exp} + 3.1MeV,\\
m_{D^0} &\approx 1.843\,GeV = m_{D^0}^{exp} - 22\,MeV.
\end{aligned} \tag{6.36}$$

By introducing a non-vanishing θ_u, we have increased $\hat r_H$ from .57 (see (5.78)) to $\hat r_H > .95$ and gotten a quite satisfactory agreement for the masses of neutral pseudoscalar mesons. So, while the need of a rather large condensate for heavy quarks is confirmed, staying with $\hat b_X < 1$ has become "much more possible" than when imposing $\theta_u = 0$.

From (5.41), using (6.12), since $b_\Omega, \hat b_\Omega \ll 1$, $\hat b_H = 1 \approx b_X$, and taking, like for calculation the masses of π^0, K^0, D^0, $\hat b_X \approx .8$, one gets

$$\boxed{\hat v_H \approx v_X \approx 151\,GeV} \tag{6.37}$$

6.4 The Higgs spectrum

From the values of the b parameters and of δ that we have determined in subsection 6.2.3 we get

$$\boxed{m_{\hat H^3} \approx 3.24\,GeV \approx m_{X^0}, \quad m_{H^0} \approx 1.65\,GeV, \quad m_{\Omega^0} \approx 86\,MeV} \tag{6.38}$$

$\hat\Omega^3$ is, like Ω^0, presumably very light, and so are Ξ^0 and $\hat\Xi^3$, for the same reasons as when $\theta_u = 0$ since we still have $b_\Xi = 0 = \hat b_\Xi$. With the value $\hat b_X = .8$ that we used to fit $m_{\pi^0}, m_{K^0}, m_{D^0}$ in subsection 6.3.2, one gets $m_{\hat X^3} \approx 2.9\,GeV$. To precisely determine $\hat b_\Omega$ an extensive study of the fermionic sector is needed.

6.5 Quark condensates

Let us have finally an estimate of the ratios of some of the quark condensates like we did in section 5.10 for the case $\theta_u = 0$. From (6.24) and the values of the b's given in (6.15) and (6.16), one gets

$$\boxed{r_H \equiv \frac{<\bar c c + \bar s s>}{<\bar u u + \bar d d>} \approx 7.42, \quad r_\Omega \equiv \frac{1}{\sqrt 2}\frac{<\bar u c + \bar c u + \bar d s + \bar s d>}{<\bar u u + \bar d d>} \approx .455} \tag{6.39}$$

$<\bar c c>$ and $<\bar s s>$ are given by the l.h.s of (5.88), which requires the knowledge of $\hat r_H$. Taking the value $\hat r_H = .96$ that we used to fit the masses of neutral pseudoscalars leads to

$$\boxed{<\bar c c> \approx 5.96\,\mu_X^3, \quad <\bar s s> \approx 4.60\,\mu_X^3} \tag{6.40}$$

which are both large negative values (we recall that $\mu_X^3 \equiv (<\bar u u + \bar d d>)/\sqrt 2$ is known from the GMOR relation). This confirms that large condensates for heavy quarks are wished for, which may me the sign that a 3rd generation is needed. These large $<\bar q q>$'s are certainly the sign that our extension of the GSW model is still incomplete.

[2] that we may identify with that of π^+ since we did not introduce electromagnetism and we know that the $\pi^+ - \pi^0$ mass difference is essentially electromagnetic.

6.6 Conclusion for the case $\theta_d \neq 0, \theta_u \neq 0$

6.6.1 Generalities

The values of θ_d and θ_u agree with the estimates $\theta_c \approx \theta_d \simeq \sqrt{\frac{|m_d|}{m_s}}$ and $\theta_u \simeq \sqrt{\frac{m_u}{m_c}}$ that were obtained on various other grounds (see footnote 4). They are independent quantities, and the Cabibbo angle $\theta_c = \theta_d - \theta_u$ cannot incorporate all the physics for 2 generations. This is in sharp contrast with the genuine GSW model in which only the difference $\theta_d - \theta_u$ is physically relevant.

Dealing with a non-vanishing θ_u provided several improvements to the fit between this model and experimental data. Together with charged pseudoscalar mesons, the masses of neutral pseudoscalars π^0, K^0, D^0 can now also be described with a good accuracy.

b_H has increased to $\approx .28$ while b_Ω and, presumably $\hat{b}_\Omega$, too, have become very small. This agrees with the fact that, at least perturbatively, non-diagonal quark condensate, which only occur at 2-loops, should be very small.

Leptonic decays of charged pions and kaons can be correctly accounted for; this goes with $b_\Omega \rightarrow 0$, which was also wished for when $\theta_u = 0$ but could not be, then, argued for.

δ has increased from $\delta \approx m_{D_s}^2$ up to $\delta \approx 5.26 GeV^2$; the mass of the "standard-like" Higgs boson has gone from $\sqrt{2} m_{D_s} = 2.78\, GeV$ up to $\sqrt{2\delta} = 3.24\, GeV$. This scaling factor concerns all the Higgs bosons. The Higgs spectrum has been modified accordingly. The 2 Higgs bosons Ω^0 and $\hat{\Omega}^3$, which had intermediate masses for $\theta_u = 0$, are now very light.

6.6.2 Rapidly varying functions, slow-converging expansions, coincidences and fine-tuning

One among the most important issues is certainly the important role of the very small parameter θ_u, concerning in particular the spectrum of the Higgs bosons. As I will show below, it is the consequence of the presence of rapidly varying functions, which often have poles, of the extreme care with which the chiral limit must be implemented ...and some "bad luck" which positioned the solution, in the case $\theta_u = 0$, inside a very special set of values of the parameters.

The value $\delta \approx m_{D_s}^2$ at $\theta_u = 0$ had been obtained by considering the s quark mass at the chiral limit, limit at which, unfortunately, we also mentioned that b_H has no reliable expansion. b_H depends of δ and can vary very rapidly with θ_c. The combination of the two values obtained for δ and θ_c determined, then, b_H to be very close to 0, but also, as shown below, close to a region where it varies rapidly. It was thus extreme and unstable fine tuning. On Fig.6.4 below we plot b_H at $\theta_u = 0$ as a function of θ_c, for $\delta \approx m_{D_s}^2$ (blue, corresponding to our result at $\theta_u = 0$) and for $\delta \approx 5.26\, GeV^2$ (red, corresponding to our result (6.14) at $\theta_u \neq 0$). The 2 vertical lines are drawn – at the experimental value (1.6) $\theta_c = .2276$; – at $\theta_c \approx .2669$ as determined from charged pseudoscalar mesons at $\theta_u = 0$ in (5.3). We see on Fig.6.4 that the values of $\theta_c = \theta_d$ and δ that were obtained at $\theta_u = 0$ coincide with $b_H \approx 0$, but that b_H is also in a domain where it varies very fast. Keeping the same value $\theta_c = .2669$ and still staying at $\theta_u = 0$, we see that varying δ between $m_{D_s}^2$ and $5.26\, GeV^2$ triggers relatively large variations of b_H. One also sees on the same figure that keeping δ fixed and varying θ_c between .228 and .267, which is not a big variation, also triggers large variations of b_H.

b_X is found to be very stable, but this is not the case for b_Ω, that behaves in many respects like b_H.

We plot on Fig.6.5 b_Ω at $\theta_u = 0$ as a function of θ_c. The 2 vertical lines are at the same positions as in Fig.6.4. For $\theta_c = .2669$ one recovers $b_\Omega \approx .2$ that we had found at $\theta_u = 0$, but going to a larger δ and / or to a smaller θ_c seems to increase b_Ω instead of bringing it close to 0 as we found for $\theta_u \neq 0$. b_Ω is therefore very sensitive to the value of the small parameter θ_u itself. This is shown on Fig.6.6 in which we plot b_Ω as a function of θ_u at the experimental value of θ_c (1.6) and at the value (6.14) of δ that we have determined. We witness the same phenomenon as the one that occurred for b_H at $\theta_u = 0$ as a function of θ_c: b_Ω is very close to 0 but it is also in a

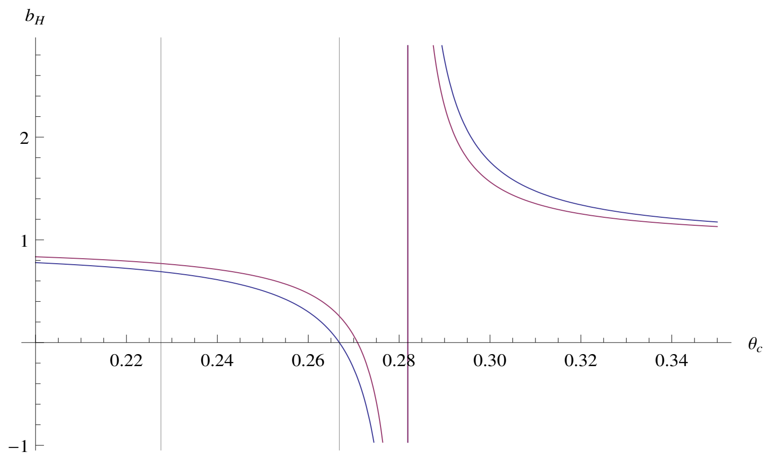

Figure 6.4: b_H at $\theta_u = 0$ as a function of θ_c for $\delta = m^2_{D_s} \approx 3.87495\, GeV^2$ (blue) and $\delta = 5.26\, GeV^2$ (red) ; the vertical lines stand at $\theta_c = .227591$ (experimental value) and $\theta_c = .2669$ (value found at $\theta_u = 0$)

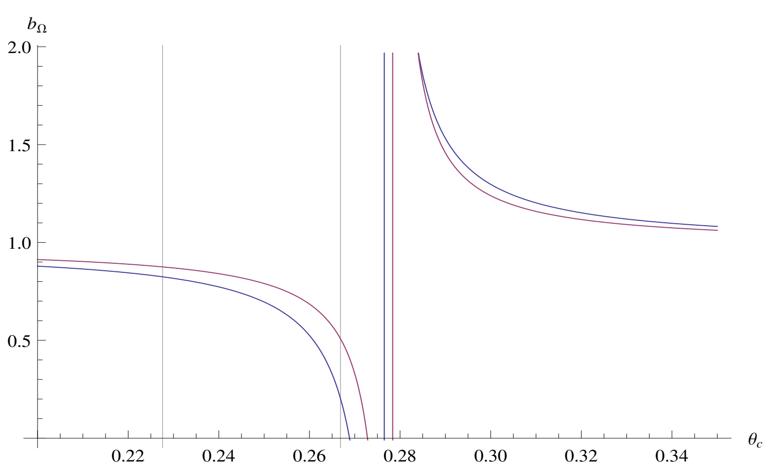

Figure 6.5: b_Ω at $\theta_u = 0$ as a function of θ_c for $\delta = m^2_{D_s} \approx 3.87495\, GeV^2$ (blue) and $\delta = 5.26\, GeV^2$ (red); the vertical lines stand at $\theta_c = .227591$ (experimental value) and $\theta_c = .2669$

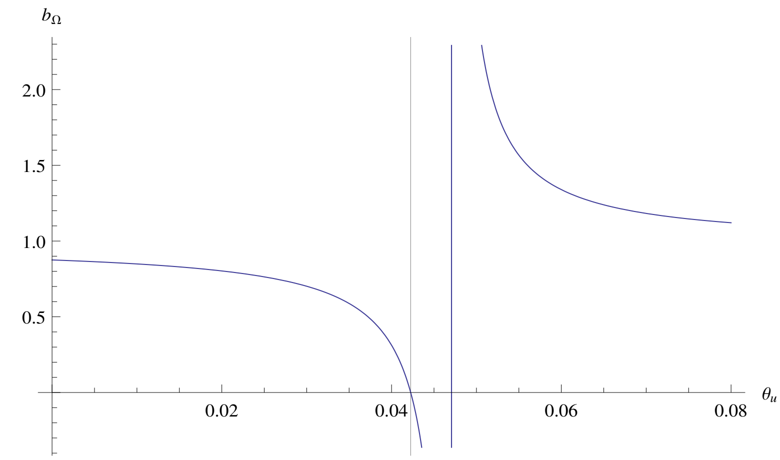

Figure 6.6: b_Ω at the measured value $\theta_c = .227591$ and $\delta = 5.529\, GeV^2$ as a function of θ_u; the vertical line stands at $\theta_u = .04225$

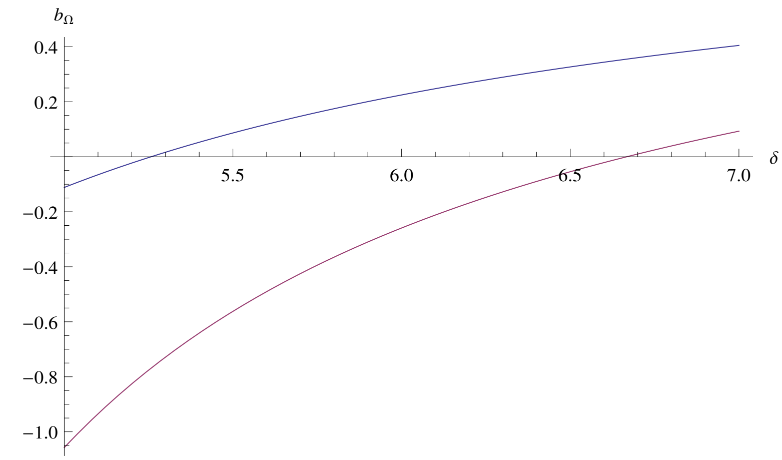

Figure 6.7: b_Ω at the measured value $\theta_c = .227591$ as a function of δ for $\theta_u = .04225$ (blue) and $\theta_u = .04436$ (red)

region of very fast variation with θ_u, close to the pole. The value of δ at which b_Ω vanishes is also very sensitive to the value of θ_u. This is seen on Fig.6.7, in which we plot b_Ω as a function of δ for two values of θ_u, $\theta_u = .04225$ as given by (6.1) (blue curve) and $1.05\times$ this value, $\theta_u = .04436$ (red curve).

Since b_Ω lies in a domain where it is very sensitive to the value of other parameters, the question is of course "can we trust the value that we have obtained?" We have no answer except that it corresponds to our intuition of non-diagonal quark condensates being very small. We recall indeed that b_Ω is related to $r_\Omega = \frac{<\bar u c + \bar c u + \bar d s + \bar s d>}{\sqrt{2} <\bar u u + \bar d d>}$ by the relation (5.15) $r_\Omega = \sqrt{b_\Omega(1-b_\Omega)/b_X(1-b_X)}$ and, since $b_X(1-b_X)$ is small, for r_Ω to be small one needs b_Ω to be very small.

We deal with a very tightly entangled series of parameters; if one changes θ_u by a very small amount (which is very conceivable because it has been deduced from the value of θ_c, which has itself experimental uncertainties, and from the masses of charged pseudoscalars, which have also some small uncertainties), one can change δ by a large amount since it roughly corresponds to the value at which b_Ω vanishes. This has in turn consequences on b_X (small, because it is very stable) and on b_H. Since large variations have been triggered by going from $\theta_u = 0$ to $\theta_u \neq 0$, it is of course necessary to take our results with care. It is unfortunate that it is also related with the existence and properties of very light scalars, one of the most interesting but also controversial domain of research of the last decades [22] [25].

We are undoubtedly dealing with very fine tuned physics, in which some parameters furthermore stubbornly resist being expanded in powers of m_π (chiral limit) or of other small parameters like θ_d or θ_u.

6.6.3 A promising way

This path of investigation was initiated [16] because it looked the most natural and, so far, the physics of pseudoscalar mesons has been remarkably well described. That there exists one and, seemingly, only one solution to the sets of equations that match experimental data gives a fair prejudice that it provides an optimized set of parameters to fit the physical world, and good confidence that it is a good way to proceed.

We are still clearly far from investigating the whole domain of even of the small subset of pseudoscalar mesons. Later works will consider in particular their semi-leptonic decays, for example the one of π^+ into a very light Higgs boson and leptons, because it could be a way to detect such an elusive particle.

While, for $\theta_u = 0$, the determination of the unique mixing angle $\theta_d = \theta_c$ and of the b parameters was made ab initio, this was not the case when we introduced the two mixing angles θ_u and θ_d. We instead relied on the measured value of $\theta_c = \theta_d - \theta_u$ to get the value of θ_u. The consistency of the method has nevertheless proved to be very satisfying.

When dealing with 2 mixing angles, we did not go either through the analysis of the fermionic constraints. The reason for this is that they can no longer be simplified and easily handled. In particular they now also include the parameter $\delta_{\Omega\bar\Omega}$, such that a larger set of equations is needed. It may well be that one is obliged to solve the whole system, which is not an easy task.

A solution to the puzzling "maximal" fermionic mixing that occurred at $\theta_u = 0$ (see section 5.8) looks nevertheless in sight. We mentioned indeed, that, at $\theta_u = 0$, the value of the Cabibbo angle that we obtained coincided with a pole of $\frac{\mu_{ds}}{\mu_d - \mu_s}$, unless b_Ω and $\hat b_\Omega$ become extremely small. We have seen that this is precisely what happens when turning on θ_u, which, despite all the issues concerning fine tuning, gives now reasonable hope for a matching between bosonic and fermionic mixing angles.

What is the final sign of the mass of the d quark? Without an exhaustive study of the whole system of equations we have unfortunately no definitive answer to give. We have seen that it should be negative for 1 generation and for 2 generations at $\theta_u = 0$. If it turns positive when switching on θ_u, this may look more "conventional", but this it also provides one more example of a "parameter" which is extremely sensitive to the small parameter θ_u. Fine tuning is then still more severe.

We did not seriously analyze the η meson, in particular concerning its orthogonality with $K^0 + \bar K^0$. Its definition itself maybe in cause and the mixing between π^0 and η must probably be taken into account.

A fairly large condensate of heavy quarks seems consistently needed. This goes against common intuition. However, as we shall study more deeply in the forthcoming work(s), quark condensation is presumably linked to the existence of strong "interactions" of quarks. They of course interact with gauge bosons (gluons), but, in this extension of the GSW model, with all scalar (Higgs) and pseudoscalar bosons, and it is rather easy to realize that their couplings are mostly not "standard" and that some of them are strong, which may provide another origin for quark condensation.

Last, doubts remain concerning the existence and spectrum of very light scalars. The reason for this is that, as shown in subsection 6.6.2, it is a highly fined tuned sector. Furthermore, this issue cannot undoubtedly be settled without dealing without including the 3rd generations and all the paraphernalia of their mixing angles (6 + CP violating phases). To add to the complexity, one knows that it is unfortunately very dangerous to approximate anyone by 0. This results in a very large system of equations to be solved simultaneously, which goes beyond the limits of this work.

Chapter 7

Symmetries. Outlook and prospects

This last chapter is mainly dedicated to the symmetries that underlie this model, and to some remarks concerning the normalization of states that pave the way to forthcoming works. We finally list the topics to be investigated in the future.

7.1 Symmetries

7.1.1 Entangled symmetries and breakings

Inside the chiral $U(4)_L \times U(4)_R$ group, we have identified in (2.11) the 3 generators of the left-handed group $SU(2)_L$ of weak interactions. Before being considered as a local group of symmetry, this $SU(2)_L$ and its mirror group $SU(2)_R$ build up the chiral group $SU(2)_L \times SU(2)_R$. In the 4-dimensional space of the 4 components $(\Delta^0, \Delta^3, \Delta^+, \Delta^-)$ of any quadruplet Δ, endowed with the corresponding basis

$$v^0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; v^3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; v^+ = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; v^- = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{7.1}$$

the three T_L generators write

$$T^3_L = \left(\begin{array}{cc|cc} 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ \hline 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{array}\right), \quad T^+_L = \left(\begin{array}{cc|cc} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ \hline -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad T^-_L = \left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{array}\right); \tag{7.2}$$

the three T_R generators write

$$T^3_R = \left(\begin{array}{cc|cc} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ \hline 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{array}\right), \quad T^+_R = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ \hline \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad T^-_R = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{array}\right); \tag{7.3}$$

the three $SU(2)$ generators $\vec{T} = \vec{T}_L + \vec{T}_R$ of the diagonal subgroup are accordingly

$$T^3 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) = Q, \quad T^+ = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad T^- = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right), \tag{7.4}$$

This diagonal $SU(2)$ is the so-called custodial $SU(2)$ group. Its 3rd generator, T^3, can be identified with the electric charge [16].

Since by anticommutation the T's close on the identity matrix $\mathbb{I}$, it is often natural to consider the larger chiral group $U(2)_L \times U(2)_R$.

In section 4.7 we also identified an "generation" $U(2)^g_L \times U(2)^g_R$ group of transformations that moves inside the 8-dimensional space of quadruplets. Its generators commute with the ones of the gauge/gauge group, such that the two groups of transformations are orthogonal. The equivalent of the custodial $SU(2)$ for horizontal transformations can be identified with the $SU(2)^g$ "generation group of symmetry". It acts as follows on quarks

$$\begin{aligned} &L^+.(c,s) = (u,d), \quad L^-.(u,d) = (c,s), \\ &L^+.(u,d) = (0,0) = L^-.(c,s), \\ &L^3.(u,d) = \frac{1}{2}(u,d), \quad L^3.(c,s) = -\frac{1}{2}(c,s). \end{aligned} \tag{7.5}$$

and is accordingly seen to be connected to flipping the generations.

The vertical and horizontal $U(2)_L \times U(2)_L$ intersect along the chiral $U(1)_L \times U(1)_R$, which, at the level of generators, are directly connected with parity (see subsection 7.1.3).

Which $SU(2)$ can be identified with the "flavor" $SU(2)$? Since it should survive for 1 generation so as to keep the 3 pions quasi-degenerate, it cannot be the generation diagonal $SU(2)$ which, as we have seen, shrinks to the trivial diagonal $U(1)$ for 1 generation. It can accordingly only be the custodial $SU(2)$.

We already emphasized in the core of the paper that, at the same time as the various symmetries are tightly entangled, so are their breaking. While the breaking of the gauge symmetry is signaled by non-vanishing bosonic VEV's $v_i, \hat{v}_i \neq 0$, fermionic condensates $< \bar{q}_i q_j > \neq 0$ are usually considered to trigger chiral symmetry breaking. We have shown, for example in (5.14), that the 2 sets of VEV's are indeed not independent.

It is worth investigating, inside each quadruplet, which subgroup of the chiral $U(4)_L \times U(4)_R$ group is left unbroken by its vacuum. We have only to focus on the diagonal $U(4)$ since we know in advance that parity is broken. A 4×4 generator of the diagonal $U(4)$ annihilates the vacuum if and only if it commutes with the corresponding $\mathbb{M}$ matrix (see (2.16) and (2.17)). Along these steps, one gets the following result: the 4 states of the type Δ^0_i are annihilated by generators of the type $\left(\begin{array}{cc|cc} a & & c & \\ & a & & c \\ \hline d & & b & \\ & d & & b \end{array}\right)$ while the 4 states of the type $\hat{\Delta}^3_i$ are annihilated by generators of the type $\left(\begin{array}{cc|cc} a & & & \\ & a & & \\ \hline & & b & \\ & & & b \end{array}\right)$. This shows that the set of 4 $< \Delta^0_i > \neq 0$ break the chiral $U(4)_L \times U(4)_R$ down to the diagonal $U(2)$ (which contains in particular $U(1)_{em}$) while the set of 4 $< \hat{\Delta}^3_i > \neq 0$ break it down to $U(1) \times U(1)_{em}$, in which the first $U(1)$ is the trivial one. In the presence of all types of VEV's, we thus conclude that the only group that is left unbroken is $U(1) \times U(1)_{em}$. Of course this paves a straightforward way for the introduction of the photon as a gauge particle.

The generators that we displayed above annihilate sets of vacuum states. If one considers a single state, for example X^0 or H^0, all generators of the type $\left(\begin{array}{cc|cc} a & & e & \\ & b & & f \\ \hline g & & c & \\ & h & & d \end{array}\right)$, with 8 independent real entries annihilate them. Likewise, $\hat X^3$ gets annihilated by $\left(\begin{array}{cc|cc} a & & & \\ & b & & f \\ \hline & & c & \\ & h & & d \end{array}\right)$ while $\hat H^3$ is annihilated by $\left(\begin{array}{cc|cc} a & & e & \\ & b & & \\ \hline g & & c & \\ & & & d \end{array}\right)$, both having only 6 independent entries. The last four are annihilated by generators with 8 independent entries (they include generators of the diagonal "generation" group $SU(2)^g$ with generators $\vec L$): Ω^0 is annihilated by $\left(\begin{array}{cc|cc} a & b & c & d \\ b & a & d & c \\ \hline e & f & g & h \\ f & e & h & g \end{array}\right)$, Ξ^0 by $\left(\begin{array}{cc|cc} a & b & c & d \\ -b & a & -d & c \\ \hline e & f & g & h \\ -f & e & -h & g \end{array}\right)$, $\hat\Omega^3$ by $\left(\begin{array}{cc|cc} a & b & c & d \\ b & a & -d & -c \\ \hline e & f & g & h \\ -f & -e & h & g \end{array}\right)$, and $\hat\Xi^3$ by $\left(\begin{array}{cc|cc} a & b & c & d \\ -b & a & d & -c \\ \hline e & f & g & h \\ f & -e & -h & g \end{array}\right)$. We see that, while for individual quadruplets the little groups can be quite large (8 dimensional), their intersection is only $U(1) \times U(1)_{em}$. In particular the chiral "generation" group $U(2)^g_L \times U(2)^g_R$ gets totally broken.

Since the chiral/gauge $SU(2)_L \times SU(2)_R$ groups are of special relevance to us, let us determine their action on the vacuum inside each quadruplet. From the general results above, we see that all Δ_i^0 states are annihilated by the diagonal $U(2)$ while all $\hat\Delta^3$ states are only annihilated by $U(1) \times U(1)_{em}$. Leaving aside the trivial $U(1)$, the situation concerning the custodial group can be summarized by

$$\vec T^{\pm}.(X^0, H^0, \Omega^0, \Xi^0) = 0, \quad T^3.(X^0, H^0, \Omega^0, \Xi^0, \hat X^3, \hat H^3, \hat\Omega^3, \hat\Xi^3) = 0. \tag{7.6}$$

$SU(2)_L$, which is to become local, is totally broken, which yields 3 Goldstone bosons inside each quadruplet. There cannot be more, since any Higgs cannot be a Goldstone, such that, even when the breaking is stronger, like for example in $\hat X^3$ or $\hat H^3$ where the chiral $U(4)_L \times U(4)_R$ with 32 generators gets broken down to a set of 6 independent generators, only 3 Goldstones are generated.

Last, for the diagonal $SU(2)^g$ group with generators $\vec L$:

$$L^3.(X^0, \hat X^3, \hat H^3, H^0) = 0, \quad L^1.(\Omega^0, \hat\Omega^3) = 0, \quad L^2.(\Xi^0, \hat\Xi^3) = 0. \tag{7.7}$$

The invariance of the scalar potential and Yukawa Lagrangian are worth some remarks. Would all normalizing factors $v_i/\sqrt{2}\mu_i^3, \hat v_i/\sqrt{2}\hat\mu_i^3$ be identical, V in (2.26) would be invariant by the whole chiral group $U(4)_L \times U(4))_R$ acting according to (2.12) on quarks. This invariance is broken because the normalizations are different. The only invariance left is the chiral $SU(2)_L \times SU(2)_R$ because the quadruplets are complex doublets of both left and right $SU(2)$. This is why this chiral/gauge group and its breaking is the most important in this kind of physics. When one evokes "chiral symmetry breaking" one generally thinks of this chiral group; its breaking by quark condensates down to the custodial $SU(2)$ generates 3 Goldstones inside each quadruplet which, for 1 generation, are the pions.

The Yukawa Lagrangian (2.23) is by construction invariant by $SU(2)_L$ but, because $\delta_{i\hat{i}} \neq \kappa_{\hat{i}i}$, it is not invariant by the chiral $SU(2)_L \times SU(2)_R$. This is why it is fitted to give soft masses to the Goldstones of the broken chiral symmetry which are not the 3 Goldstones of the broken gauge symmetry.

7.1.2 Left or right?

Another important point is that such an extension of the GSW model can be a right-handed $SU(2)_R$ as well as left-handed $SU(2)_L$ gauge theory. This because all quadruplets that we have built are stable by both groups and are isomorphic to complex $SU(2)_L$ by the laws of transformations (2.18) and/or $SU(2)_R$ doublets by the laws of transformation (2.20).

Then, why not a $SU(2)_L \times SU(2)_R$ spontaneously broken gauge theory? For 1 generation, the answer is clear: with only 4 pseudoscalar and 4 scalars one has not enough degrees of freedom to provide at the same time 6 Goldstones and 3 physical pions: the latter should vanish from the spectrum, together with the η and 2 charged scalars. However this argument is no longer valid for more generations. We have already mentioned the fact that η being a Goldstone is very unphysical in the case of 1 generation, but that for 2 generations, the role is played by another diagonal pseudoscalar meson etc ... The only necessity is to provide 6 true Goldstones that vanish from the spectrum. For 2 generations is seems hard to achieve, but this may be kept in mind for 3 generations, where in particular the physics of $\bar{q}(\gamma_5)t$ bound states, with expected strong coupling, could be very rich (and at the same time difficult to handle).

At this stage one can only state that, by its scalar structure, the extension that we propose has the potential to also be extended to a left-right gauge theory.

7.1.3 Parity and its breaking

The action of the $U(1)_L$ and $U(1)_R$ *generators* $\mathbb{I}_L$ and $\mathbb{I}_R$ on the quadruplets is interesting since, like for 1 generation, it is directly connected with parity. Indeed, for any pair $\Delta_i, \hat{\Delta}_i, i \in [X, H, \Omega, \Xi]$ of quadruplets, one has, like for 1 generation (see (2.22)),

$$\mathbb{I}_L . \frac{\sqrt{2}\mu_i^3}{v_i}\Delta_i = -\frac{\sqrt{2}\hat{\mu}_i^3}{\hat{v}_i}\hat{\Delta}_i, \quad \mathbb{I}_R . \frac{\sqrt{2}\mu_i^3}{v_i}\Delta_i = +\frac{\sqrt{2}\hat{\mu}_i^3}{\hat{v}_i}\hat{\Delta}_i. \tag{7.8}$$

The group $U(1)_L \times U(1)_R$ is a subgroup of both the vertical and horizontal $U(2)_L \times U(2)_R$ chiral groups. Its diagonal subgroup is the trivial $U(1)$ with generator the 4×4 identity matrix $\mathbb{I}$, which is just multiplying ψ by a phase.

One of the most conspicuous aspect of parity violation is that the 2 charged longitudinal $W_{\|}^{\pm}$ are scalars while the neutral $W_{\|}^3$ is pseudoscalar.

Would parity be an unbroken symmetry, the VEV's of parity transformed doublets $< \mathfrak{s}^0 + \mathfrak{p}^3 >$ and $< \mathfrak{p}^0 + \mathfrak{s}^3 >$ would be identical, which is not the case.

7.1.4 The generation group of transformations

We are going here to give more information about the chiral generation group $SU(2)_L^g \times SU(2)_R^g$ with generators $\vec{L}$ given in (4.26).

The action of the generators of its diagonal subgroup on quarks has been given in (7.5).

At the level of bilinear quark operators, using the laws of transformations (2.14), one gets the following results (we forget below about the normalizations of the quadruplets).

By the action of $SU(2)^g_L$, the 8 quadruplets split into the 4 following doublets

$$\begin{pmatrix} X-\hat{X} \\ \Omega-\hat{\Omega}+(\Xi-\hat{\Xi}) \end{pmatrix}, \quad \begin{pmatrix} \Omega-\hat{\Omega}-(\Xi-\hat{\Xi}) \\ H-\hat{H} \end{pmatrix}, \quad \begin{pmatrix} H+\hat{H} \\ \Omega+\hat{\Omega}+(\Xi+\hat{\Xi}) \end{pmatrix}, \quad \begin{pmatrix} \Omega+\hat{\Omega}-(\Xi+\hat{\Xi}) \\ X+\hat{X} \end{pmatrix}, \tag{7.9}$$

while by $SU(2)^g_R$ they split into the 4 following ones

$$\begin{pmatrix} X+\hat{X} \\ \Omega+\hat{\Omega}+(\Xi+\hat{\Xi}) \end{pmatrix}, \quad \begin{pmatrix} \Omega+\hat{\Omega}-(\Xi+\hat{\Xi}) \\ H+\hat{H} \end{pmatrix}, \quad \begin{pmatrix} H-\hat{H} \\ \Omega-\hat{\Omega}+(\Xi-\hat{\Xi}) \end{pmatrix}, \quad \begin{pmatrix} \Omega-\hat{\Omega}-(\Xi-\hat{\Xi}) \\ X-\hat{X} \end{pmatrix}, \tag{7.10}$$

which contain states of mixed $1\pm\gamma_5$ parity. In both (7.9) and (7.10) the upper and lower components have respectively $+\frac{1}{2}$ and $-\frac{1}{2}$ quantum numbers with respect to L^3_L and L^3_R. One moves inside each doublet by the action of the raising/lowering operators L^+, L^-. Moving inside the whole 8-dimensional space of quadruplets requires sequences of alternate left and right transformations.

With respect to the diagonal $SU(2)^g$ the 8 multiplets split into 2 singlets and 2 triplets which contain states of given parity

$$(X+H), \quad (\hat{X}+\hat{H}), \quad \begin{pmatrix} \Omega+\Xi \\ X-H \\ \Omega-\Xi \end{pmatrix}, \quad \begin{pmatrix} \hat{\Omega}+\hat{\Xi} \\ \hat{X}-\hat{H} \\ \hat{\Omega}-\hat{\Xi} \end{pmatrix}. \tag{7.11}$$

It should be clear that, in (7.9), (7.10) and (7.11) above, we are dealing with multiplets of quadruplets.

The generation group reproduces therefore, in the space of quadruplets, the symmetry pattern that the chiral/gauge group created inside each quadruplet: doublets for left and right groups, singlet + triplet for the diagonal group.

In (7.11), let us remark that the neutral pseudoscalar entries of $\Omega\pm\Xi$, $\hat{\Omega}\pm\hat{\Xi}$ entries match $K^0\pm\bar{K}^0$ and $D^0\pm\bar{D}^0$, but that their charged pseudoscalar entries are $K^\pm\pm D^\pm$. This is to be contrasted with $\Omega, \Xi, \hat{\Omega}, \hat{\Xi}$ which separate charged D and K mesons but mix the neutral ones.

7.2 The mass pattern of neutral scalars

Unlike pseudoscalar mesons which are built from quark mass eigenstates, these states are by construction flavor $\bar{q}_i q_j$ eigenstates.

Despite our lack of information concerning the Ξ^0 and $\hat{\Xi}^3$ Higgs bosons, they are presumably light and, at least from perturbative arguments, quasi-degenerate. So, the mass pattern of neutral scalars found in section 6.4 clearly exhibits a splitting into 1 heavy triplet $(\hat{H}^3, X^0, \hat{X}^3)$, 2 light doublets $(\Xi^0, \hat{\Xi}^3)$, $(\Omega^0, \hat{\Omega}^3)$ and 1 singlet H^0 with intermediate mass. We also notice that, going from $\theta_u = 0$ to $\theta_u \neq 0$, 2 mass scales have been swapped, but that the structure into 1 triplet, 2 doublets and 1 singlet has been seemingly preserved. Some $SU(2)$ symmetry can therefore be suspected to be at work.

One one side, this comes as a surprise because of the apparent anarchy in the spectrum of scalar mesons that appears in experimental data [20]. On the other side, there should a priori be no reason why $SU(2)$ or $SU(3)$ *"rotated"* flavor symmetry operates on pseudoscalar $\bar{q}^i_m\gamma_5 q^j_m$ bound states of quark mass eigenstates while their unrotated scalar partners keep insensitive to a similar symmetry. A noticeable difference is however that this result came out only after rather long calculations and could hardly have been guessed from the start.

The question that arises is the nature of this symmetry. As far as pseudoscalar mesons are concerned it is usual to consider that it is the diagonal-custodial $SU(2)$. Indeed, for 1 generation, the breaking of $SU(2)_L\times SU(2)_R$ down to the diagonal $SU(2)$ is invoked to yield the 3 pions as Goldstone bosons. It is comforted by the fact that the

4 components of any quadruplet split into a neutral singlet plus a triplet of the custodial $SU(2)$, which perfectly fits the pion case; the 3 states correspond to the quantum numbers $(-1, 0, +1)$ of T^3. $T^{\pm}$ are the raising-lowering operators that move inside the pion triplet. For 1 generation, the diagonal flavor $SU(2)$ coincides with the strong $SU(2)$ of isospin which operates, at the quark level, in the (u, d) space.

As far as scalar mesons are concerned, the situation looks more intricate. Indeed, the components of the doublets and the triplet into which they split belong to different quadruplets: $(X, \hat{X}, \hat{H})$ for the triplet, $(\Omega, \hat{\Omega})$ and $(\Xi, \hat{\Xi})$ for the doublets. So, obviously, pairs of parity-transformed quadruplets are involved, and, accordingly, the chiral $U(1)_L \times U(1)_R$ symmetry which we have shown to be tightly related to parity plays a role. Would the quasi-standard Higgs $\hat{H}^3$ be in reality a singlet and its degeneracy with X^0 and $\hat{X}^3$ purely accidental, one could think that the 3 doublets include pairs of bosons belonging to parity transformed quadruplets, which one could interpret as a left over of parity symmetry. But this does not fit H^0 and $\hat{H}^3$ which seem largely split, at least for 2 generations.

The solution may lie in (7.7). While *the* vacuum of the theory is defined by the 8-set of VEV's $\{< X^0 >, < \hat{X}^3 >, < H^0 >, < \hat{H}^3 >, < \Omega^0 >, < \hat{\Omega}^3 >, < \Xi^0 >, < \hat{\Xi}^3 >\}$, the effective scalar potential V_{eff} that we minimize to get the Higgs masses is split into 4 terms involving only pairs of parity-transformed quadruplets. We expect accordingly that the symmetry group in relation with any such pair of Higgs bosons is the one leaving invariant the corresponding part of V_{eff}. The unbroken subgroup corresponds therefore to the generators that annihilate the relevant pair of vacuum states: L^1 for $(\Omega, \hat{\Omega})$, L^2 for $(\Xi, \hat{\Xi})$, L^3 for both $(X, \hat{X})$ and $(H, \hat{H})$. The Higgs bosons are then expected to split into $SU(2)^g$ multiplets labeled by the quantum numbers of L^1, L^2 and L^3. For the first two pairs, they can be only doublets (or singlets), for the last two pairs, the corresponding 4 Higgs bosons could split into singlets, doublets of triplet. Our calculations show that they seemingly fall into 1 singlet + 1 triplet. Therefore, the mass pattern of Higgs boson appears as the left imprint of the broken generation symmetry.

Now, why such a mass pattern has not been detected?

For the 3 heaviest states, because they correspond to 3 quasi-degenerate Higgs bosons at the same mass as the "quasi-standard" Higgs. First, for 2 generations, it is not yet heavy enough and our results are non-physical, yet. Secondly, as we shall see in the next work, the 2 partners of $\hat{H}^3$ are much more weakly coupled and therefore hard to detect.

As far as the lightest scalars are concerned, as we shall also show in a subsequent work, they are also certainly very difficult to detect because they are lighter than the pions, extremely weakly coupled to leptons. Furthermore, they are plagued with non-perturbative couplings to hadronic matter, which makes theoretical predictions hazardous.

One is left with 1 neutral scalar at an intermediate mass, around $1.6\,GeV$. Though the spectrum will certainly be modified when a 3rd generation is added, one can already notice that some states like f^0 of K^{*0} fall in this mass range.

One must also be careful not to draw too fast conclusions because the mass pattern that we have obtained may also be the consequence of our choice for the scalar potential and for the Yukawa Lagrangian. We recall that we did not consider the most general, but, for the latter, we wanted to cancel from the start FCNC's, and for the former we wished to satisfy general requirements concerning, in particular chiral symmetry. The most general expressions would include a dramatically large number of parameters, trigger many unwanted processes, and their systematic treatment would certainly be, in practice, unfeasible. So, maybe we have only shown that such a construction is feasible, and that it leads to interesting consequences that have not been obtained in other frameworks. After all, it nature is not simple, what can we do ?

7.3 Outlook

7.3.1 Miscellaneous remarks

• This work is nothing more than a fit of (a small part of) the physical world at low energy (pseudoscalar mesons, gauge bosons) by an extension of the GSW model which is the smallest one that can naturally incorporate all known mesons. It was not evident from the start that a solution existed and could be found rather simply, but it does exist, and we have reasonable hope that the very few problems which are left (orthogonality between some neutral mesons, large quark condensates) will find a solution when one more generation is added. Arguments that led, in the past, to the introduction of heavy fermions and technicolor-like theories [2] become void; heavy fermions are not needed and enough energy scales and VEV's are available to describe low energy physics up to the electroweak scale.

• The cornerstone of the whole construction is the one-to-one correspondence demonstrated in section 2.1 between the complex Higgs doublet of the GSW model and some very specific quadruplets of bilinear quark operators that mix states of different parities (scalars and pseudoscalars). This set of quadruplets split into two parity transformed subsets of the type $(\mathfrak{s}, \vec{\mathfrak{p}})$ and $(\mathfrak{p}, \vec{\mathfrak{s}})$. This type of configuration was never investigated before but comes out naturally as soon as the bijection is uncovered. In particular, even for 1 generation, 2 Higgs doublets are necessary, which are parity-transformed of each other. For N generations, this number grows to $2N^2$.

• The underlying operating symmetries are all subgroups of the chiral $U(2N)_L \times U(2N)_R$. Parity has been tightly linked to the chiral $U(1)_L \times U(1)_R$ group, which is broken to the trivial diagonal $U(1)$. A "generation" $SU(2)^g_R \times SU(2)^g_L$ group has also been identified, which operates in the space of quadruplets; its diagonal part flips generations at the quark level.

• The Higgs bosons are composite states that fit into the family of $J = 0$ mesons built with 4 quarks. They exhibit quasi-degeneracies, which appear as the imprint of the broken symmetry among generations. The mass of the "quasi-standard" Higgs boson increases like that of the heaviest pseudoscalar meson. This puts it at the order of the D_s scale for 2 generations.

• Naturalness.
The point of view that we adopt is that a theory is natural if and only if it describes nature as we observe it. Our extension of the GSW model has been tailored to this purpose. The only assumptions that have been made concern the scalar potential and Yukawa Lagrangian. Since they do not spoil the description of physical reality, we consider them as natural. More general potentials of Yukawa Lagrangian are conceivable, but at the price of loosing much elegance and simplicity.

• Light Higgses and dark matter.
An important phenomenon is the appearance of light scalars. This shows that there are other motivations as the ones exposed for example exposed in [21] to look for such particles. It is also a highly fine tuned sector. Only a careful study of their couplings can tell whether they can have escaped detection up to now. If yes, they role as one of the possible components of dark matter [22] should of course be carefully scrutinized.

• Quark condensation and strongly coupled light scalars.
Light scalars happen to be extremely weakly coupled to everything except to hadronic matter, to which they are strongly coupled [18]. This is likely (see 7.3.2) to be the signal of the existence of massive hadronic bound state(s). Then, their normalization plays a crucial role in their physical coupling to the rest of the world. The mass of hadronic bound states, if we think for example of pions, goes along with chiral symmetry breaking and quark condensation, phenomena commonly attributed to strong interactions, that is, in the absence of no other candidate, to gluons. Now, in the case where quarks get also strongly coupled to light scalars, it is worth investigating whether

this could be another source of vacuum instability and quark condensation. We have of course specially in mind the condensation of heavy quarks, which seem to be needed.

7.3.2 A prelude to forthcoming works : nature and normalization of asymptotic states

After the whole set of equations is solved, we can in principle calculate all couplings that occur in the Lagrangian as functions of physical masses and measured parameters. However, the *physical* couplings, in particular between mesons (which includes Higgs bosons) and quarks, crucially depend on the normalization of the former.

To make this easily understandable, let us take once again (see section 1.1) the example of the charged pion, which occurs inside the X quadruplet (4.1). The kinetic terms (4.10) for $X^\pm$ are normalized to 1 and the corresponding Yukawa couplings, given by (2.23), include $\delta_X \ X^- \left(\frac{v_X}{\sqrt{2}\mu_X^3}\sqrt{2}\bar{u}\gamma_5 d\right) + \ldots$ with $\sqrt{2}\mu_X^3 \ =< \bar{u}u + \bar{d}d >$. $\delta_X = \delta(1-b_X)$ by (2.31), $v_X \approx 151\,GeV$ and μ_X^3, given by the GMOR relation, close to $\sqrt{2}f_\pi^2 m_{\pi^+}^2/(m_u+m_d)$. This yields, using (6.12) a coupling $\delta\frac{b_X(1-b_X)\hat{v}_H(m_u+m_d)}{\sqrt{2}f_\pi^2 m_\pi^2} X^-\bar{u}\gamma_5 d \approx -29 \ X^-\bar{u}\gamma_5 d$ which we do not know a priori how to handle [1]. More information is however available from Current Algebra because X^- is connected to the charged pion by (1.3) and that, accordingly, the Yukawa coupling rewrites $\delta_X \frac{v_X}{f_\pi}(\cos\theta_c \pi^- + \ldots)\frac{v_X}{\sqrt{2}\mu_X^3}(\cos\theta_c\sqrt{2}\bar{u}_m\gamma_5 d_m + \ldots)$ inducing a coupling of π^- to quarks equal to $\delta(1-b_X)\frac{v_X^2}{f_\pi\mu_X^3}\cos^2\theta_c \pi^- \bar{u}_m\gamma_5 d_m$. It is also $\gg 1$, but we did not yet take into account the normalization of the charged pions. The kinetic terms (4.10) for the X Higgs multiplet induce indeed pion kinetic terms $\partial_\mu(\frac{v_X}{f_\pi}\cos\theta_c\pi^-)\partial^\mu(\frac{v_X}{f_\pi}\cos\theta_c\pi^+)$ and normalizing the pions back to 1 amounts to dividing the kinetic + Yukawa Lagrangian by $v_X^2\cos^2\theta_c/f_\pi^2$. This rescales the coupling between charged pions and quarks to $\delta(1-b_X)\frac{f_\pi}{\mu_X^3}\pi^-\bar{u}_m\gamma_5 d_m \approx \frac{\delta(1-b_X)(m_u+m_d)}{\sqrt{2}f_\pi m_\pi^2}\pi^- \ \bar{u}\gamma_5 d \approx -.018\,\pi^= \bar{u}\gamma_5 d$ which is much smaller (in modulus) that 1 [2]. The *physical* coupling of pion to quarks is smaller than 1, but we could only calculate it with the help of low energy relations.

This simple example shows that, if the asymptotic bosonic states are not correctly identified (like we did first, considering that $X^\pm$ are asymptotic states), or if they are not suitably normalized (like we did in the second case when we ignored the normalization of the pions), their couplings to quarks that occur in the genuine Yukawa Lagrangian are not the physical ones. The normalization plays a specially important role because the kinetic terms (4.10) are quadratic in the mesonic fields while the genuine Yukawa couplings (2.23) are linear.

It can therefore occur that the physical coupling of a meson to quarks can be much smaller than the one naively read on the Yukawa Lagrangian. For pions, luckily enough we know the normalization factor from elementary Current Algebra considerations, but, for scalar mesons, the situation is more intricate [3].

7.4 Prospects

Very natural composite extensions of the GSW model exist, with no extra fermion, which are very likely to suitably describe known physics up to the electroweak scale. At least they provide an optimal set of parameters to fit basic physics from the chiral scale up to the electroweak scale. The Higgs spectrum is much richer than in the GSW model and exhibits quasi-degeneracies. It the trend observed for 1 and 2 generations persists, the mass of the quasi-standard Higgs boson will continue to grow like that of the heaviest pseudoscalars. A general phenomenon

[1] Of course, it is also fine-tuned and depends on how close b_X is to 1.

[2] In the simple case of 1 generation (see chapter 3) $\theta_c = 0$, $\delta_X = m_\pi^2$ and $v_X = f_\pi$: the normalization factor v_X/f_π is 1 and the Yukawa couplings exactly match the mass term $m_\pi^2\pi^+\pi^-$ for charged pions (fulfilling the PCAC and GMOR low energy relations), to which only the X quadruplet contributes. For 2 generations, Yukawa couplings rewrite in their bosonised form $(\frac{v_X\cos\theta_c}{f_\pi})^2\delta(1-b_X)\pi^+\pi^-$ while kinetic terms rewrite $(\frac{v_X\cos\theta_c}{f_\pi})^2\partial_\mu\pi^+\partial^\mu\pi^-$; $\sqrt{\delta(1-b_X)} \approx .134\,GeV$ is now only close to the pion mass since the pion receives contributions to its mass from several quadruplets.

[3] The estimates of the decays $V \to \gamma + light\ scalar$ in [24] have to be re-investigated since "standard" Yukawa couplings between scalars and quarks have been used. At the end of its paper, Wilczek warns that it is only valid in the strict framework of the GSW model with a light Higgs boson.

seems also to be the occurrence of light neutral scalars. All this makes the Higgs sector a privileged place to look for new physics [18]. Since, for example, W's and quarks get their masses from several Higgs bosons, their couplings cannot be standard any more. It is in particular important to re-investigate in this framework the so-called "decoupling theorem" [23]. It is indeed disquieting that, in the GSW model, very heavy fermions, which should behave like quasi-classical objects, do no decouple and even have large quantum effects (loops), which manifest themselves, for example, in the decay $Higgs \to \gamma\gamma$. At the same time, the GSW model should be recovered in a suitable limit (probably, according to what we learned up to now, at the price of a lot of fine tuning).

For many reasons, it is highly desirable to introduce the 3rd generation and to look for a solution of the system of equations that would determine the 18 bosonic VEV's, 18 fermionic VEV's, 3×18 Yukawa couplings, more than 6 mixing angles, the masses of the 18 Higgs bosons *etc*. This is a very hard task, all the more as none of the 6 mixing angles (forgetting CP violating phases) can safely be turned to 0. The example of 2 generations has moreover shown, by successive trials and errors, that these equations can only be solved in a carefully devised order, and that no global / numerical solution can be trusted. Reasons for this is that the equations involve fast varying functions of potentially complex variables, with poles here and there which make calculations highly unstable, and, on a general ground, that solving consistently a system of more than 30 equations including trigonometric functions is highly non-trivial. It looks therefore hopeless to solve them like we did for 2 generations.

We end this incomplete work by an (also incomplete) list of goals to achieve:
* find the spectrum of Higgs bosons for 3 generations and all their couplings; confirm in particular that one of them weights around 125 GeV;
* confirm (or infirm) the presence of light scalars;
* find reasonable arguments for a large $< \bar{t}t >$ condensate, presumably in relation with the strong coupling of light Higgses to quarks;
* have a closer look at the η (and so ...) mesons and their orthogonality to other neutral pseudoscalars;
* calculate all mixing angles in terms of bosonic and fermionic quantities; check the matching (or not) between the two sets of relations;
* include CP violating effects;
* investigate whether and how light scalars could be detected, actualize the corresponding calculation of a vector particle V decaying into γ + a light scalar [24];
* re-analyze the data of radiative J/Ψ and Υ decays [4] [25];
* study these light scalars as possible components of dark matter [22];
* study their roles in other phenomena, like the 1/2 leptonic decays of π^+, the $g-2$ of the muon [26] or the proton radius [27];
* after getting the physical couplings of quarks to gauge and to Higgs bosons, revisit the limit of very heavy fermions [23] and compare with the GSW model;
* recover the GSW model in a certain limit.

Acknowledgments: *It is a pleasure to thank V.A. Novikov and M.I. Vysotsky, who always ask the right questions at the right time. I also want to thank S. Davidson, G. Moultaka and P. Slavich for their friendly and constructive remarks.*

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.1 Appendix : Flavor quark bilinears expressed in terms of quark mass eigenstates

We write below the formulæ expressing the flavor bilinear quark operators in terms of bilinears of quark mass eigenstates and of the 2 mixing angles θ_u and θ_d. They are largely used throughout the paper. The formulæ for scalar and pseudoscalar bilinears are of course similar.

$$\begin{aligned}
&* \, \bar u\gamma_5 d = c_u c_d\, \bar u_m\gamma_5 d_m + c_u s_d\, \bar u_m\gamma_5 s_m + s_u c_d\, \bar c_m\gamma_5 d_m + s_u s_d\, \bar c_m\gamma_5 s_m,\\
&* \, \bar u\gamma_5 u - \bar d\gamma_5 d = \frac{c_u^2+c_d^2}{2}(\bar u_m\gamma_5 u_m - \bar d_m\gamma_5 d_m) + \frac{c_u^2-c_d^2}{2}(\bar u_m\gamma_5 u_m + \bar d_m\gamma_5 d_m)\\
&\qquad + \frac{s_u^2+s_d^2}{2}(\bar c_m\gamma_5 c_m - \bar s_m\gamma_5 s_m) + \frac{s_u^2-s_d^2}{2}(\bar c_m\gamma_5 c_m + \bar s_m\gamma_5 s_m)\\
&\qquad + s_u c_u(\bar u_m\gamma_5 c_m + \bar c_m\gamma_5 u_m) - s_d c_d(\bar d_m\gamma_5 s_m + \bar s_m\gamma_5 d_m)\\
&* \, \bar u\gamma_5 u + \bar d\gamma_5 d = \frac{c_u^2+c_d^2}{2}(\bar u_m\gamma_5 u_m + \bar d_m\gamma_5 d_m) + \frac{c_u^2-c_d^2}{2}(\bar u_m\gamma_5 u_m - \bar d_m\gamma_5 d_m)\\
&\qquad + \frac{s_u^2+s_d^2}{2}(\bar c_m\gamma_5 c_m + \bar s_m\gamma_5 s_m) + \frac{s_u^2-s_d^2}{2}(\bar c_m\gamma_5 c_m - \bar s_m\gamma_5 s_m)\\
&\qquad + s_u c_u(\bar u_m\gamma_5 c_m + \bar c_m\gamma_5 u_m) + s_d c_d(\bar d_m\gamma_5 s_m + \bar s_m\gamma_5 d_m).
\end{aligned} \tag{12}$$

$$\begin{aligned}
&* \, \bar c\gamma_5 s = s_u s_d\, \bar u_m\gamma_5 d_m - s_u c_d\, \bar u_m\gamma_5 s_m - c_u s_d\, \bar c_m\gamma_5 d_m + c_u c_d\, \bar c_m\gamma_5 s_m,\\
&* \, \bar c\gamma_5 c - \bar s\gamma_5 s = \frac{c_u^2+c_d^2}{2}(\bar c_m\gamma_5 c_m - \bar s_m\gamma_5 s_m) + \frac{c_u^2-c_d^2}{2}(\bar c_m\gamma_5 c_m + \bar s_m\gamma_5 s_m)\\
&\qquad + \frac{s_u^2+s_d^2}{2}(\bar u_m\gamma_5 u_m - \bar d_m\gamma_5 d_m) + \frac{s_u^2-s_d^2}{2}(\bar u_m\gamma_5 u_m + \bar d_m\gamma_5 d_m)\\
&\qquad - s_u c_u(\bar u_m\gamma_5 c_m + \bar c_m\gamma_5 u_m) + s_d c_d(\bar d_m\gamma_5 s_m + \bar s_m\gamma_5 d_m),\\
&* \, \bar c\gamma_5 c + \bar s\gamma_5 s = \frac{c_u^2+c_d^2}{2}(\bar c_m\gamma_5 c_m + \bar s_m\gamma_5 s_m) + \frac{c_u^2-c_d^2}{2}(\bar c_m\gamma_5 c_m - \bar s_m\gamma_5 s_m)\\
&\qquad + \frac{s_u^2+s_d^2}{2}(\bar u_m\gamma_5 u_m + \bar d_m\gamma_5 d_m) + \frac{s_u^2-s_d^2}{2}(\bar u_m\gamma_5 u_m - \bar d_m\gamma_5 d_m)\\
&\qquad - s_u c_u(\bar u_m\gamma_5 c_m + \bar c_m\gamma_5 u_m) - s_d c_d(\bar d_m\gamma_5 s_m + \bar s_m\gamma_5 d_m).
\end{aligned} \tag{13}$$

$$\begin{aligned}
&* \, \bar u\gamma_5 s + \bar c\gamma_5 d = -s_{u+d}(\bar u_m\gamma_5 d_m - \bar c_m\gamma_5 s_m) + c_{u+d}(\bar u_m\gamma_5 s_m + \bar c_m\gamma_5 d_m),\\
&* \, (\bar u\gamma_5 c + \bar c\gamma_5 u) - (\bar d\gamma_5 s + \bar s\gamma_5 d) = c_{2u}(\bar u_m\gamma_5 c_m + \bar c_m\gamma_5 u_m) - c_{2d}(\bar d_m\gamma_5 s_m + \bar s_m\gamma_5 d_m)\\
&\quad - \frac{s_{2u}+s_{2d}}{2}(\bar u_m\gamma_5 u_m - \bar d_m\gamma_5 d_m) - \frac{s_{2u}-s_{2d}}{2}(\bar u_m\gamma_5 u_m + \bar d_m\gamma_5 d_m) + s_{2u}\bar c_m\gamma_5 c_m - s_{2d}\bar s_m\gamma_5 s_m,\\
&* \, (\bar u\gamma_5 c + \bar c\gamma_5 u) + (\bar d\gamma_5 s + \bar s\gamma_5 d) = c_{2u}(\bar u_m\gamma_5 c_m + \bar c_m\gamma_5 u_m) + c_{2d}(\bar d_m\gamma_5 s_m + \bar s_m\gamma_5 d_m)\\
&\quad - \frac{s_{2u}-s_{2d}}{2}(\bar u_m\gamma_5 u_m - \bar d_m\gamma_5 d_m) - \frac{s_{2u}+s_{2d}}{2}(\bar u_m\gamma_5 u_m + \bar d_m\gamma_5 d_m) + s_{2u}\bar c_m\gamma_5 c_m + s_{2d}\bar s_m\gamma_5 s_m.
\end{aligned} \tag{14}$$

$$\begin{aligned}
&* \, \bar u\gamma_5 s - \bar c\gamma_5 d = s_{u-d}(\bar u_m\gamma_5 d_m + \bar c_m\gamma_5 s_m) + c_{u-d}(\bar u_m\gamma_5 s_m - \bar c_m\gamma_5 d_m),\\
&* \, (\bar u\gamma_5 c - \bar c\gamma_5 u) - (\bar d\gamma_5 s - \bar s\gamma_5 d) = (\bar u_m\gamma_5 c_m - \bar c_m\gamma_5 u_m) - (\bar d_m\gamma_5 s_m - \bar s_m\gamma_5 d_m),\\
&* \, (\bar u\gamma_5 c - \bar c\gamma_5 u) + (\bar d\gamma_5 s - \bar s\gamma_5 d) = (\bar u_m\gamma_5 c_m - \bar c_m\gamma_5 u_m) + (\bar d_m\gamma_5 s_m - \bar s_m\gamma_5 d_m).
\end{aligned} \tag{15}$$